Tangent Bundle on Three Dimension Small Cover

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Abstract: Small covers arising from 3-dimensional simple polytopes. The geometry of the tangent bundle of a three dimension small cover is an old topic. The theory is of great importance in mathematics and physics. It is an interesting question to understand whether tangential bundles exist on the 3-dimensional small cover. In this paper a three dimension small cover on the tangent bundle is defined by using a spin structure on the compact and oriented smooth three manifold. The SO(3) is diffeornorphic to the unit tangent bundle of the two-sphere. Since the SO(3) is diffeornorphic to RP3. The line bundle over RP3 is just trivial bundle. We use fiber bundle pullbacks measures prove that 3-dimensional small cover has a trivial tangent bundle. To the knowledge of the author of this paper, we only considered the special case on small cover and effectively connected Clifford algebras, spin groups, and small cover. Note that the general case is much more complicated. Some results in the general case were not proved.

Keywords: Small Cover; Tangent Bundle; Fiber Bundle Pullbacks

1. Introduction

Toric topology emerged as a new mathematical discipline atncross roads of algebraic topology,combinatorics ,algebraic geometry,commutative algebra and symplectic geometry in tha last decades of the 20th century.It studies spaces with toric actions whose orbit spaces have nice combinatorial structures such as simple polytopes and simplicial complexes. Small cover was firstly introduced in [1]. An n-dimensional small cover is n-dimensional smooth manifold M^n adimtting a Z_2^n -action and its orbit space is an n-dimensional simple convex polytope . We can construct a small cover $M^n(P^n,\lambda)$ over P^n. In [2] , Shintar $\hat{o}$ KUROKI and Zhi $L\ddot{u}$,have been studied the projective bundles over small covers .In [3], L.Wu and L.Yu caculated the Fundamental groups of small covers. In [4], Erokhovets built an explicite decomposition of 3 dimensional small covers into geometric parts.Many further works has been carried out (see in [5-9]). In ,we find the conection between small cover and spin geometry. In section 2,we rewrite the small cover and tangent bundle. In section 3, we introduce clifford algebras and spin groups. In section 4,we recall spin structures on the three dimensional small cover .In section 5,we recall the tangent bundles on group SO(3).In section 6,we use use fiber bundle pullbacks measures prove that three dimensional small cover has a trivial tangent bundle $TM^n(P^n,\lambda)$.

2. Small Cover and Tangent Bundle

Definition 2.1 Setting P^n is an n-dimensional convex polyhedron, and if each vertex of P^n has n codimensions of 1, we call this convex polyhedron is single convex polyhedron.

Definition 2.2 Function $\lambda : F \to Z_2^n$ is called characteristic function, if you meet the following conditions: any the vertices, have $|\lambda(F_1),\lambda(F_2),.....,\lambda(F_n)| = \pm 1$.

The tangent bundle is a geometric construction that associates each point on a manifold with its corresponding tangent space. It is a fundamental concept in differential geometry and allows for the study of vectors and vector fields on the manifold.

Formally, given a smooth manifold M , the tangent bundle of M , denoted as TM , is defined as the disjoint union of the tangent spaces at each point in M . Mathematically, we express this as:

$TM = \cup_{P\in M} T_P M$ .Here, $T_P M$ represents the tangent space at point p in the manifold M, and the disjoint union $\cup$ represents the collection of all tangent spaces at each point. In simple terms, TM comprises all possible tangent vectors at each point of M.

The elements of TM are called tangent vectors or vector fields. A tangent vector at a given point p on the manifold can be thought of as an arrow or a directional derivative associated with that point. The tangent bundle combines all these arrows or directional derivatives into a cohesive structure that covers the entire manifold.

The tangent bundle TM itself is a manifold of twice the dimension of the original manifold M. It inherits a natural topology from M, allowing for the study of smooth vector fields on the manifold. This topology enables the definition of smooth functions, differentiation, integration, and other geometric operations on the tangent bundle.

3. Clifford Algebras and Spin Groups

Definition3.1 Let V be an n-dimensional real vector space with an inner product $\langle .,. \rangle$ and choose an orthonormal basis $e_1, e_2, \ldots, e_n$. Associated to V is the real Clifford algebra C(V).This is a 2^n -dimensional real vector space and an algebra wit unit 1.It is generated by the basis vectors $e_1, e_2, \ldots, e_n$ with multiplication rules $e_i^2 = -1, e_i e_j = -e_j e_i$ for $i \neq j$.

Definition3.2 (three involutions) Let $n \in N$,we have the following involutions:

(1) $C_n = C(R^n, -x_1^2 - \ldots - x_n^2) -$ Clifford algebra of the n-dimensional real negative definite form.

(2) $C'_n = C(R^n, x_1^2 + \ldots + x_n^2) -$ Clifford algebra of the n-dimensional real positive definite quadratic form.

(3) $C'^c_n = C(C^n, z_1^2 + \ldots + z_n^2) -$ Clifford algebra of the n-dimensional complex quadratic form.

Definition3.3 $Pin(n) \subset C_n$ is the group which is multiplicatively generated by all vectors $x \in S^{n-1}$. Therefore ,the elements of $Pin(n)$ are the products $x_1, \ldots, x_m$ with x $x_i \in R^n, \|x_i\| = 1$,the spin group , $Spin(n)$,is defined as $Spin(n) = Pin(n) \cap C_n^0$.

Proposition 3.1 Let $n \in N$,the map $\lambda_n : Spin(n) \to SO(n)$ defined by $\forall_{x \in spin(n)} \forall_{v \in R^n} \lambda_n(x)v = xv\bar{x}$ is a continuous group epimorphism with kernel equal to $\{\pm 1\}$. Moreover for $n \geq 3$ Spin(n) is simply connected and λ_n is the universal covering of SO(n).

4. Spin Structures on the Three Dimension Small Cover

Proposition4.1 Each smooth, compact, 3-manifold Y exists a spin structure.

Proof :Let $\Sigma \subset Y$ be a 2-dimensional submanifold and let be $\vee_\Sigma$ the normal bundle .

Let

$$W_2(T\Sigma) = \chi(\Sigma)(\text{mod}2), W_2(V_\Sigma)$$
$$= \Sigma \cdot \Sigma(\text{mod}2), \chi(\Sigma) = W_1(T\Sigma)^2(\text{mod}2)'$$

We have

$$\langle W_2(TY), [\Sigma] \rangle = W_2(T\Sigma) + W_1(T\Sigma) \cdot$$
$$W_1(V_\Sigma) = \chi_2(\Sigma) + W_1(T\Sigma)^2 = 0,$$

where χ_2 be the mod-2 Euler characteristic . Finally,we have $W_2(TY) = 0$.

Proposition4.2 An orientable closed manifold x has a spin structure if and only if its second Stiefel-Whitney class vanishes $W_2(x) = 0$ .Moreover in this case spin structures on x are classified by $H^1(x, z_2)$.

Following the above proposition ,we know that an orientable closed three dimension small cover have a spin structure.

5. Tangent Bundles on Group SO(3)

The unit tangent bundle of the two-dimensional sphere is diffeornorphic to the SO(3). The first two columns receive an orthogonal matrix from the diffeomorphism. The 3-sphere serves as the universal cover of SO(3) because it is diffeomorphic to the unit tangent bundle of the 2-sphere. The tangent space of group SO(3) is the Lie algebra. The group SO(3)'s tangent bundles may then be obtained.

Defition 5.1 Let x be an orientable closed manifold of dimension n.Let Q be its principal SO(n)-principal bundle.A spin structure on x is a pair(P,^),such that P is a principal

Spin(n)-bundle cover x and $\wedge : P \to Q$ is 2-fold covering for which the following diagram commutes, as shown in Figure 1:

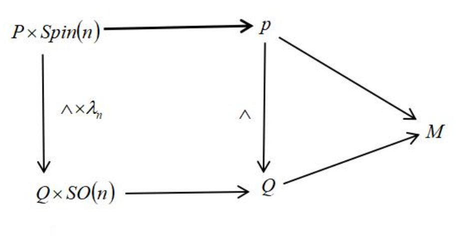

Figure 1. Tangent Bundles on Group SO(3)

Where the defined by the action of the group Spin(n) and SO(n) on the principal bundles P and Q respectively.The map $\lambda_n : Spin(n) \to SO(n)$ is a continuous group epimorphism with kernel equal to $\{\pm 1\}$. Moreover for $n \geq 3$ Spin(n) is simply connected and λ_n is the universal covering of SO(n).

6. Tangent Bundles Pullbacks

Let $\widetilde{f} : TQ \to TM$ is is a continuous.Where TQ is the tangent bundle of group SO(3).

Since the group SO(3) is naturally diffeornorphic to RP^3 . The line bundle over RP^3 is just trivial bundle.So we get there are just trivial tangent bundle on oriented three dimensional small cover . Therefor ,the Stiefel-Whitney over oriented three dimensional small cover with spin stucture is trivial .Since $w(RP^n) = 1$ if and only if $n = 2^r - 1 (r \geq 0)$ .So when $n = 3$, $w(RP^3) = 1$ .As shown in Figure 2.

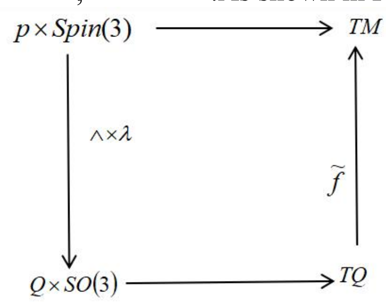

Figure 2. Tangent Bundles Pullbacks

7. Conclusion

In this paper, we have proved that three dimensional small cover has a trivial tangent bundle $TM^n(P^n, \lambda)$ .The existence of a trivial tangent bundle enables us to better understand the topological structure and characteristics of Small Cover. By studying the properties of the tangent bundle, we can uncover important information about the topological dimension, connectivity, and singular point distribution of Small Cover. This is highly beneficial for comprehending and classifying Small Cover as well as the properties of related topological spaces.

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