Exploration of Calculation Methods for a Class of Sparse Determinants

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Abstract: Claw-shaped determinants, as a type of sparse determinant, have extensive applications in various fields such as matrix theory, numerical computation, physics, engineering, economics, and computer science. To efficiently compute the values of claw-shaped determinants, this paper first categorizes them into four types. Next, using the properties of determinants, the calculation formulas for these four types of claw-shaped determinants are discussed. Finally, numerical examples demonstrate the rationality of these formulas.

Keywords: Determinant; Claw-Shaped; Sparsity; Laplace's Theorem

1. Introduction

In many practical applications, such as image processing, signal processing, scientific computing, and social networks, the data being processed often takes the form of sparse matrices (matrices where the number of nonzero elements is much smaller than the total number of elements), this has made the theoretical study of sparse matrices one of the research hotspots [1-5].

The problem of calculating the determinant of sparse matrices is significant for improving the theoretical foundation of sparse matrices. For example, the computation of tridiagonal determinants plays an important role in the discretization of differential equations, linear systems of equations, and physical problems. Claw-shaped determinants are another type of determinant with a sparse structure. This sparsity not only significantly reduces computational effort when calculating their values but also minimizes storage costs. Previous researchers have studied the computation of claw-shaped determinants [6-10], but no systematic exploration of their calculation methods has been conducted. Therefore, this paper will comprehensively discuss the computation formulas for different types of claw-shaped determinants.

To facilitate the discussion, we first define four types of claw-shaped determinants as follows.

Definition 1.1 Let a_i, b_j, c_j be numbers not all equal to zero, where $i = 1, 2, \cdots, n$; $j = 2, 3, \cdots, n$. Determinants of order n:

$$D_n^1 = \begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ c_2 & b_2 & 0 & \cdots & 0 & 0 \\ c_3 & 0 & b_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_{n-1} & 0 & 0 & \cdots & b_{n-1} & 0 \\ c_n & 0 & 0 & \cdots & 0 & b_n \end{vmatrix},$$

$$D_n^2 = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_3 & a_2 & a_1 \\ 0 & 0 & \cdots & 0 & b_2 & c_2 \\ 0 & 0 & \cdots & b_3 & 0 & c_3 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & b_{n-1} & \cdots & 0 & 0 & c_{n-1} \\ b_n & 0 & \cdots & 0 & 0 & c_n \end{vmatrix},$$

$$D_n^3 = \begin{vmatrix} c_n & 0 & 0 & \cdots & 0 & b_n \\ c_{n-1} & 0 & 0 & \cdots & b_{n-1} & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ c_3 & 0 & b_3 & \cdots & 0 & 0 \\ c_2 & b_2 & 0 & \cdots & 0 & 0 \\ a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \end{vmatrix},$$

$$D_n^4 = \begin{vmatrix} b_n & 0 & \cdots & 0 & 0 & c_n \\ 0 & b_{n-1} & \cdots & 0 & 0 & c_{n-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_3 & 0 & c_3 \\ 0 & 0 & \cdots & 0 & b_2 & c_2 \\ a_n & a_{n-1} & \cdots & a_3 & a_2 & a_1 \end{vmatrix}.$$

These are referred to as the first, second, third, and fourth types of claw-shaped determinants, respectively.

For the first type of claw-shaped determinants, extensive research results already exist. Below, we list some of the known conclusions.

Lemma 1.1 Let D_n^1 be a determinant of order n of the first type of claw-shaped determinant. Then:

(i) When $\prod_{j=2}^{n} b_j \neq 0$,

$$D_n^1 = \left(a_1 - \sum_{j=2}^{n} \frac{a_j c_j}{b_j}\right) \prod_{j=2}^{n} b_j .$$

(ii) When there is exactly one zero among $b_2, b_3, \cdots, b_n$, without loss of generality, let $b_i = 0\left(i \in \{2,3,\cdots,n\}\right)$, then:

$$D_n^1 = -a_i c_i \prod_{j \neq i, j=2}^{n} b_j .$$

(iii) When at least two elements among $b_2, b_3, \cdots, b_n$ are zero,

$$D_n^1 = 0 .$$

Expanding the formula for case (i) in Lemma 1.1 reveals that the result also satisfies the conclusions in (ii) and (iii). Thus, the computation formula for the first type of claw-shaped determinant can be unified as follows:

$$D_n^1 = a_1 \prod_{j=2}^{n} b_j - \sum_{j=2}^{n} \left(a_j c_j \prod_{k \neq j, k=2}^{n} b_k \right).$$

Given the above conclusions for the first type of claw-shaped determinant, do similar conclusions exist for the other three types of claw-shaped determinants? The following sections will continue to explore these and provide computation formulas for other types of claw-shaped determinants.

2. Main Conclusions

Theorem 2.1 If the n-order determinant D_n^2 is a second-type claw determinant, then

$$D_n^2 = (-1)^{\frac{n(n-1)}{2}} \left[a_1 \prod_{j=2}^{n} b_j - \sum_{j=2}^{n} \left(a_j c_j \prod_{k \neq j, k=2}^{n} b_k \right) \right].$$

Proof When $\prod_{j=2}^{n} b_j \neq 0$, adding $-\frac{c_{n+1-k}}{b_{n+1-k}}$ times the k $(k = 1,2,3,\cdots,n-1)$ column of D_n^2 to the last column yields:

$$D_n^2 = \begin{vmatrix} a_n & a_{n-1} & \cdots & a_3 & a_2 & a_1 - \sum_{j=2}^{n} \frac{a_j c_j}{b_j} \\ 0 & 0 & \cdots & 0 & b_2 & 0 \\ 0 & 0 & \cdots & b_3 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & b_{n-1} & \cdots & 0 & 0 & 0 \\ b_n & 0 & \cdots & 0 & 0 & 0 \end{vmatrix}$$

$$= (-1)^{\frac{n(n-1)}{2}} \left(a_1 - \sum_{j=2}^{n} \frac{a_j c_j}{b_j} \right) \prod_{j=2}^{n} b_j .$$

When one element in $b_2, b_3, \cdots, b_n$ is zero, without loss of generality, suppose $b_i = 0\left(i \in \{2,3,\cdots,n\}\right)$.

Expanding D_n^2 along the i-th row gives:

$$D_n^2 = (-1)^{i+n} c_i \begin{vmatrix} a_n & \cdots & a_{i+1} & a_i & a_{i-1} & \cdots & a_2 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & b_2 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & b_{i-1} & \cdots & 0 \\ 0 & \cdots & b_{i+1} & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ b_n & \cdots & 0 & 0 & 0 & \cdots & 0 \end{vmatrix}$$

then expanding the determinant in the formula along the column containing a_i:

$$D_n^2 = (-1)^{i+n} c_i (-1)^{1+(n+1-i)} a_i \begin{vmatrix} 0 & \cdots & 0 & 0 & \cdots & b_2 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & b_{i-1} & \cdots & 0 \\ 0 & \cdots & b_{i+1} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ b_n & \cdots & 0 & 0 & \cdots & 0 \end{vmatrix}$$

$$= (-1)^{\frac{(n-2)(n-3)}{2}} a_i c_i \prod_{j \neq i, j=2}^{n} b_j .$$

When at least two elements in $b_2, b_3, \cdots, b_n$ are zero, suppose $b_i = b_j = 0$, where $i \in \{2,3,\cdots,n\}$; $j \in \{2,3,\cdots,n\}$; $i \neq j$. selecting the i, j rows, we find that all their second-order minors equal zero. By Laplace's theorem, it follows that $D_n^2 = 0$.

Now we prove that regardless of the values of $b_i\left(i = 2,3,\cdots,n\right)$,

$$D_n^2=(-1)^{\frac{n(n-1)}{2}}\left[a_1\prod_{j=2}^{n}b_j-\sum_{j=2}^{n}\left(a_jc_j\prod_{k\neq j,k=2}^{n}b_k\right)\right].$$

When $\prod_{j=2}^{n}b_j\neq 0$ and at least two of b_2, b_3, $\cdots$, b_n are zero, expand each formula to prove it. When b_2, b_3, $\cdots$, b_n has only one zero, without loss of generality, let $b_i=0\left(i\in\{2,3,\cdots,n\}\right)$. By expanding both formulas and comparing the results, it can be observed that the two formulas differ by at most a sign. Thus, it suffices to prove that $(-1)^{\frac{n(n-1)}{2}}=(-1)^{\frac{(n-2)(n-3)}{2}+1}$. Let $F(n)=\frac{n(n-1)}{2}-\left[\frac{(n-2)(n+3)}{2}+1\right]$, it must be proven that for any $n\in Z^+$, $F(n)$ is always an even number. Because

$$F(n)=\frac{n(n-1)}{2}-\left[\frac{(n-2)(n-3)}{2}+1\right]=2n-2\in 2Z.$$

Thus, the proof is complete.

$$D_n^2=(-1)^{\frac{n(n-1)}{2}}\left[a_1\prod_{j=2}^{n}b_j-\sum_{j=2}^{n}\left(a_jc_j\prod_{k\neq j,k=2}^{n}b_k\right)\right].$$

Similarly, the calculation formulas for third-type and fourth-type claw determinants can be derived, and their proofs are similar to the proof of Theorem 2.1, so they are omitted.

Theorem 2.2 If the n-order determinant D_n^3 is a third-type claw determinant, then

$$D_n^3=(-1)^{\frac{n(n-1)}{2}}\left[a_1\prod_{j=2}^{n}b_j-\sum_{j=2}^{n}\left(a_jc_j\prod_{k\neq j,k=2}^{n}b_k\right)\right].$$

Theorem 2.3 If the n-order determinant D_n^4 is a fourth-type claw determinant, then

$$D_n^4=a_1\prod_{j=2}^{n}b_j-\sum_{j=2}^{n}\left(a_jc_j\prod_{k\neq j,k=2}^{n}b_k\right).$$

3. Numerical Examples

If an n-order determinant belongs to one of the types of claw-shaped determinants, its value can be directly obtained using the corresponding computation formula. In general, the structure of an n-order determinant may not be a claw-shaped determinant. However, for certain determinants, their properties can be utilized to transform them into claw-shaped determinants. The computation formulas provided in this paper can then be used to calculate their values. Below are specific examples.

Example1 3.1 Compute the value of the *n*-order determinant:

$$D_n=\begin{vmatrix} b_1 & b_1 & \cdots & b_1 & b_1 & a_1+b_1\\ b_2 & b_2 & \cdots & b_2 & a_2+b_2 & b_2\\ b_3 & b_3 & \cdots & a_3+b_3 & b_3 & b_3\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ b_{n-1} & a_{n-1}+b_{n-1} & \cdots & b_{n-1} & b_{n-1} & b_{n-1}\\ a_n+b_n & b_n & \cdots & b_n & b_n & b_n \end{vmatrix}.$$

Solution Using the edge-addition method [11], we have:

$$D_n=(-1)^{(-n-2)}\begin{vmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 1\\ b_1 & b_1 & \cdots & b_1 & b_1 & a_1+b_1 & b_1\\ b_2 & b_2 & \cdots & b_2 & a_2+b_2 & b_2 & b_2\\ b_3 & b_3 & \cdots & a_3+b_3 & b_3 & b_3 & b_3\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ b_{n-1} & a_{n-1}+b_{n-1} & \cdots & b_{n-1} & b_{n-1} & b_{n-1} & b_{n-1}\\ a_n+b_n & b_n & \cdots & b_n & b_n & b_n & b_n \end{vmatrix}.$$

Multiply the last column of the determinant by -1 and add it to each of the preceding columns:

$$D_n=(-1)^{(-n-2)}\begin{vmatrix} -1 & -1 & \cdots & -1 & -1 & -1 & 1\\ 0 & 0 & \cdots & 0 & 0 & a_1 & b_1\\ 0 & 0 & \cdots & 0 & a_2 & 0 & b_2\\ 0 & 0 & \cdots & a_3 & 0 & 0 & b_3\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots\\ 0 & a_{n-1} & \cdots & 0 & 0 & 0 & b_{n-1}\\ a_n & 0 & \cdots & 0 & 0 & 0 & b_n \end{vmatrix}.$$

Using the computation formula for the second type of claw-shaped determinant:

$$D_n=(-1)^{\left(\frac{n^2-n-4}{2}\right)}\left[\prod_{j=1}^{n}a_j+\sum_{j=1}^{n}\left(b_j\prod_{k\neq j,k=1}^{n}a_k\right)\right].$$

Example 3.2 Compute the value of the n-order determinant:

$$D_n=\begin{vmatrix} b & b & \cdots & b & b & a\\ b & b & \cdots & b & a & b\\ b & b & \cdots & a & b & b\\ \vdots & \vdots & & \vdots & \vdots & \vdots\\ b & a & \cdots & b & b & b\\ a & b & \cdots & b & b & b \end{vmatrix}.$$

Solution Multiply the last row of the determinant D_n by -1 and add it to each of the preceding rows:

$$D_n = \begin{vmatrix} b-a & 0 & \cdots & 0 & 0 & a-b \\ b-a & 0 & \cdots & 0 & a-b & 0 \\ b-a & 0 & \cdots & a-b & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ b-a & a-b & \cdots & 0 & 0 & 0 \\ a & b & \cdots & b & b & b \end{vmatrix}.$$

Using the computation formula for the third type of claw-shaped determinant:

$$D_n = (-1)^{\frac{n(n-1)}{2}} (a-b)^{n-1} \left[a + (n-1)b\right].$$

Example 3.3 Compute the value of the n-order determinant:

$$D_n = \begin{vmatrix} a & b & b & \cdots & b & b \\ b & a & b & \cdots & b & b \\ b & b & a & \cdots & b & b \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & \cdots & a & b \\ b & b & b & \cdots & b & a \end{vmatrix}.$$

Solution Multiply the last row of the determinant D_n by -1 and add it to each of the preceding rows:

$$D_n = \begin{vmatrix} a-b & 0 & 0 & \cdots & 0 & b-a \\ 0 & a-b & 0 & \cdots & 0 & b-a \\ 0 & 0 & a-b & \cdots & 0 & b-a \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a-b & b-a \\ b & b & b & \cdots & b & a \end{vmatrix}.$$

Using the computation formula for the fourth type of claw-shaped determinant:

$$D_n = (a-b)^{n-1}\left[a + (n-1)b\right].$$

4. Conclusion

Claw-shaped determinants are a special type of determinant characterized by sparsity. This paper leverages their structural characteristics to not only provide a rigorous classification of different types of claw-shaped determinants but also derive computation formulas for each type. Numerical examples demonstrate that the computation formulas presented in this paper facilitate more convenient and accurate calculation of the values of various types of claw-shaped determinants.

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References

[1] Jarmila Jancarik, Sung-Hou Kim. Sparse Matrix Sampling: A Screening Method for Crystallization of Proteins. Journal of Applied Crystallography, 1991, 24(4): 409-411.

[2] Hackbusch Wolfgang. A Sparse Matrix Arithmetic Based on H-Matrices. Part I: Introduction to H-Matrices. Computing, 1999, 62(2): 89-108.

[3] Mingrui Yang, Shibing Zhou, Qian Wang, et al. Fast Multi-view Clustering of Sparse Matrices and Improved Normalized Cuts. Computer Science and Exploration, 2024, 18(11): 3027-3040.

[4] Samuel Williams, Leonid Oliker, Richard Vuduc, et al. Optimization of Sparse Matrix–vector Multiplication on Emerging Multicore Platforms. Parallel Computing, 2009, 35(3): 178-194.

[5] Johann Walter Kolar, Martin Baumann, Frank Schafmeister, et al. Novel Three-phase AC-DC-AC Sparse Matrix Converter. APEC. Seventeenth Annual IEEE Applied Power Electronics Conference and Exposition, 2002, 2: 777-791.

[6] Ruiling Jia, Mingjuan Sun. A Brief Analysis of Determinant Types and Their Computation Methods. Mathematics Learning and Research, 2020, (8): 16-18.

[7] Yawen Li, Caiyun Liu. Three Special Determinants and Their Generalized Computation Methods. Science Enthusiast (Education and Teaching), 2020, (4): 247-248.

[8] Lanyun Bian. A Brief Discussion on Determinant Computation Methods. Mathematics Learning and Research, 2017, (5): 20.

[9] Liqiang Chen. Completion of Solutions for Claw-shaped Determinants. China Market, 2015, (14): 199-200+206.

[10]Jingxiao Zhang, Dejie Jiao, Shuxia Kong. Calculation of "Claw-shaped" and "Cross-shaped" Determinants. Hebei Science Teaching Research, 2006, (4): 56-58.

[11]Efang Wang, Shengming Shi. Advanced Algebra. Beijing: Higher Education Press, 2019.