Dissipativity Analysis of General Linear Methods for Coupled Systems of Nonlinear Functional Differential and Functional Equations

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Abstract: The analysis of long-term behavior in nonlinear systems is one of the core contents in modern applied mathematics. Its key characteristic is dissipativity, which implies that the system exhibits a certain form of global boundedness or the ultimate decay of energy, ensuring that the system will eventually enter a globally attracting set. Studying dissipativity is of great significance for analyzing the stability and controllability of nonlinear functional systems. Currently, functional differential equations depend not only on the current state but also on past states. When coupled with other types of equations, dissipativity analysis becomes relatively difficult and extremely complex. For such coupled systems, it is necessary to analyze their numerical methods and identify appropriate research tools.

Keywords: Nonlinear Functional Differential and Functional Equations; Coupled Systems; General Linear Methods; Dissipativity

1. Introduction

In most cases, general linear methods are employed to handle the numerical integration framework through multi-step and multi-stage methods, which can maintain the dissipativity of the original system at the numerical level—this has become a practical key in dissipativity analysis. From a teaching perspective, students need to understand not only the definition of dissipativity and the construction of general linear methods but also develop a unique mathematical modeling mindset that transitions from continuous to discrete systems. When analyzing numerical methods, the ability to preserve the geometric properties of the system is required, which involves integrating multiple fields such as numerical solutions of differential equations and dynamic systems to achieve cross-domain analysis. This places higher demands on students' foundational knowledge and abstract thinking. Traditional teaching approaches that proceed from definitions and theorems to proofs often yield poor results; therefore, continuous exploration of teaching strategies is imperative for current development.

2. Analysis of Core Difficulties and Breakthrough Strategies

In the practical process of conducting dissipativity analysis for coupled systems of nonlinear functional equations, it is necessary to accurately identify the pain points and difficulties encountered by students during learning and formulate corresponding breakthrough strategies. The common difficulties include the following.

2.1 Relatively Abstract Concepts

Concepts such as functional differential equations and infinite-dimensional dynamic systems make it difficult for students to fully understand ordinary differential equations, let alone establish intuitive mental models. To address this, breakthrough strategies should involve intuitive analogy-based teaching. For example, through real-life analogies, functional differential equations can be compared to systems with memory capabilities. For instance, a person's past decisions do not depend solely on their current emotional state—similarly, the current rate of change of a system depends not only on its current state but also on experiences from previous days, meaning the function itself is influenced by historical states. When analyzing coupled systems, they can be analogized to a team: each member is part of the team, and their individual states directly affect the team's overall state. Similarly, in a coupled system, each subsystem exerts an influence on other subsystems.

Geometric intuition can also be used to guide dissipativity teaching. When explaining dissipativity, teachers should avoid directly presenting highly abstract mathematical definitions. Instead, they should start with the simplest autonomous ODE system ($\dot{x} = f(x)$), and help students understand by plotting vector fields and phase portraits. This enables students to fully recognize that a functional differential equation is essentially a curve of historical functions, and its space corresponds to a function space (e.g., C-space or L^2-space). This approach allows students to learn more clearly and comprehensively, thereby improving their overall learning quality.

2.2 Complex Theoretical Proofs

Dissipativity analysis involves content such as the construction of Lyapunov functionals, and the proof process is extremely lengthy, often causing students to become lost in the details of the proof. To solve this problem, teachers need to help students focus on the core theorems in the analysis, such as the dissipativity theorem for continuous systems and the dissipativity theorem for numerical methods. Before proving these theorems, teachers can use a block diagram to demonstrate the overall logical flow and how the logic is derived, enabling students to grasp the big picture first before delving into details.

For example, when explaining Lyapunov functionals, teachers should emphasize why the functional adopts the form of "the squared norm of the current state plus the integral of historical states." This helps students understand the content by controlling both the current and historical energy, simplifying the originally complex estimation process into skill-based scaling. In this way, students can focus on the purpose and feasibility of scaling, enhancing their understanding of the teaching content.

3. Teaching Design and Implementation of Core Teaching Content

To improve students' learning outcomes when studying and discussing this content, it is necessary to design progressive teaching modules based on the core teaching content outlined above.

3.1 Theoretical Foundation Module

he teaching content of this module focuses on analyzing the dissipativity of coupled systems from ordinary differential equations (ODEs) to functional differential equations (FDEs). By reviewing the dissipativity theory of ODEs and the Lyapunov function method, FDEs and their coupled systems are gradually introduced. The selection of phase spaces (e.g., C-space or L^2-space) is explained, with emphasis on introducing the one-sided Lipschitz condition as a key assumption for dissipativity analysis and providing a rigorous mathematical definition of dissipativity for coupled systems.

In practical teaching, case-driven analysis is adopted. A specific coupled system is used as an example, such as:

{ẋ(t)=f(x(t),y(t),y(t-τ)), (ODE component)

{y(t)=g(x(t),y(t), ∫ _{t- τ }^t h(y(s))ds), (functional equation component)

Teachers should guide students to verify step-by-step whether this system satisfies the one-sided Lipschitz condition and attempt to construct a simple Lyapunov functional. Visual teaching tools can also be used for assistance: MATLAB or Python are employed to plot the solution trajectories and phase portraits of simple delayed systems (e.g., the Hutchinson equation), helping students develop an initial understanding of "absorbing sets" and "global attractors."

3.2 Numerical Methods Module

The teaching content of this module covers the construction and properties of general linear methods (GLMs). The focus is on systematically introducing the formulas, structure, order conditions, and stability theory of GLMs, with key explanations of their essence as "one-step multi-stage" methods and how to interpret the stage values of Runge-Kutta (RK) methods and the historical information of linear multi-step methods.

A "Transformer-style" explanation approach is used to demonstrate from multiple perspectives how to "transform" the classic 4-stage 4th-order RK method and the 2-step Adams-Bashforth method into GLM formats. By filling in their respective (A, U, B, V) matrices, students can understand the physical meaning of the GLM coefficient matrices. A group teaching model is adopted: students are divided into groups, and instead of analyzing the content independently, they are required to gradually rewrite a familiar numerical method into a GLM form and present it in class. This approach significantly deepens their understanding of the GLM framework.

3.3 Coupling Breakthrough Module

Establishing an analytical framework for numerical dissipativity theory is the core content of analyzing the dissipativity of general linear methods for coupled systems of nonlinear functional differential and functional equations. Through this content, the algebraic conditions that enable GLMs to preserve the dissipativity of coupled systems can be derived, which typically involves concepts such as algebraic stability and diagonal stability of numerical methods.

Learning is conducted using a "three-step" derivation method:

Review of continuous problems: Write down the key inequality for the dissipativity of continuous systems ($dV/dt \leq \alpha - \beta V$).

Construction of numerical discretization: Apply the GLM format to the test system and assume that the numerical solution satisfies a similar discrete inequality ($V_{n+1} \leq (1 - \beta\Delta t)V_n + \alpha\Delta t$).

Building the bridge: Through complex calculations and scaling, prove that for the discrete inequality to hold, the coefficient matrix of the GLM must satisfy certain specific relationships (e.g., B is positive definite, A satisfies a certain condition, etc.). The details of this step can be appropriately simplified, but the logical chain must be clear.

To facilitate analysis, "mnemonics" can be refined: complex algebraic conditions are summarized into easy-to-remember "mnemonics" or "checklists." For example: "To achieve numerical dissipativity, first check if matrix B is positive definite, then verify the conditions for matrix A, and never forget the relationship between V and U."

3.4 Practical Application Module

Through numerical experiments and error analysis, students can better understand the dissipativity of general linear methods for coupled systems of nonlinear functional differential and functional equations. The focus is on helping students program and implement a specific GLM that satisfies dissipativity conditions and apply it to the coupled system example in Module 1. Long-term integration is performed to verify the boundedness of the numerical solution, and comparisons are made with formats that do not meet the conditions. Meanwhile, the long-term behavior of numerical errors is analyzed.

Project-based learning is adopted: this module is designed as a small-scale research project lasting 2–3 weeks. Students are required to complete a task book including algorithm implementation, numerical simulation, graphical presentation, result analysis, and report writing. Open-ended questions are designed to guide students: for example, "How does the size of the system's attracting set change when the coupling strength τ is modified? Can the numerical method capture this change?" and "When the step size h is too large, will even a GLM that theoretically satisfies the conditions fail? Why?" These questions aim to cultivate students' research capabilities.

4. Conclusion

In summary, the teaching and analysis of dissipativity for coupled systems of nonlinear functional differential and functional equations is an extremely challenging task. In education and teaching, a student-centered approach should be adopted to help students intuitively understand the relevant content. With numerical practice as the foundation, students' abilities are cultivated, their comprehensive problem-solving strategies are improved, and their anxiety during learning is reduced—thereby stimulating their intrinsic motivation to learn. Additionally, advanced knowledge can be gradually transformed into tangible, implementable, and investigable teaching content, helping students effectively master the knowledge related to the topic of this course and enhance their comprehensive computational literacy when facing complex scientific problems, laying a solid foundation for their future studies.

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