Stability and Convergence Analysis of Runge-Kutta Methods for a Class of Nonlinear Functional Differential-Algebraic Equations

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Abstract: In mathematics teaching, the numerical analysis course has always been one of the most important bridges connecting mathematical theory and scientific practice. It directly affects students' learning effects in mathematics learning and is even more related to students' ability and quality in solving complex engineering problems. In the traditional mathematics teaching process, the focus of education and teaching is mostly placed on the numerical teaching methods of ordinary differential equations. However, insufficient attention has been paid to cutting-edge models and nonlinear functional differential-algebraic equations. Therefore, it is necessary to analyze this content, strengthen the cultivation of theoretical foundations, help students develop critical learning thinking, integrate it into the entire knowledge system, and stimulate students' innovative awareness.

Keywords: A Class of Nonlinear Functional Differential-Algebraic Equations; Runge-Kutta Method; Stability; Convergence

1. Introduction

The application of the Runge-Kutta method is not simply a superposition of different contents. It deepens the reform of mathematics teaching and further improves the teaching content, thereby enhancing the theoretical depth, knowledge breadth, and cutting-edge nature of knowledge in the classroom. In the process of mathematics teaching, multi-dimensional optimization of teaching quality is achieved. This not only enables effective analysis of numerical content but also cultivates more outstanding and high-quality compound talents for Chinese society.

2. Stability and Convergence of Runge-Kutta Methods for a Class of Nonlinear Functional Differential-Algebraic Equations

In the process of analyzing the initial value problem, it is necessary to solve for a given FDAEs∈S（α，β，γ，σ，L1，L2，L3）with respect to a given grid $\Delta h \in \{\Delta h\}$. For any parallel integration step, （tn，ψ，x，y1，z1，y2，z2.....yn，zn）→（tn+1，ψ，x，y1，z1，y2，z2.....yn+1，zn+1）, which is composed of the following formula:

$$\begin{cases} \tilde{y}^{h}(t) = \Pi^{h}\left(t, \tilde{\psi}, \tilde{y}_{1}, \tilde{y}_{n+1}\right) \\ \tilde{Z}^{h}(t) = \Pi^{h}\left(t, \tilde{\psi}, \tilde{Z}_{1}, \tilde{Z}_{n+1}\right) \end{cases} \quad (1)$$

where $h_n \leq \bar{h}$, a constant C=4(2(α+βL1)+¯CCπ)‖bT‖ can be obtained, and L=(L1+L2Cπ+L3CπC2)(1+Chn)+L2Cπ+L3Cπ C2. Through the analysis of the above content and data, it is concluded that C, L, and h̄ depend on the Lipschitz constants α，β，γ，L1，L2，L3 and the solution method adopted.

It is assumed that the initial function is sufficiently smooth, and different symbols represent different contents. M_i and $\bar{M}_i$ respectively represent different contents. Among them, the true solutions y(t) and Z(t) of the problem are the bounds of the derivatives of various orders. M_i and $\bar{M}_i$ have moderate magnitudes, and the derivatives of various orders of y(t) and Z(t) are continuous on [0, T]. Based on this, the theoretical results of stability and convergence are established [1].

3. Application Methods of Stability and Convergence Teaching of Runge-Kutta Methods for a Class of Nonlinear Functional Differential-Algebraic Equations

3.1 Transformation from Being a User to an Understander

After explaining the stable region of the classic ordinary differential equation (ode), teachers need to actively guide students to conduct independent thinking and analyze how the stability conditions of the equation change after the addition of delay terms and algebraic constraints. In traditional mathematics teaching, students are mostly taught to use MATLAB or Python library functions as data tools. However, the stability and convergence of the R-K method for FDAEs are not fully and in-depth explained, resulting in students failing to learn the algorithm in depth and lacking an understanding of the algorithm core. Teachers should guide students to independently compare the stable regions of different R-K methods, select the classic comparison method and the specific method for FDAEs, and analyze the differences between them. This enables students to solve problems when facing different algebraic constraints and more intuitively observe the differences between the two methods. Furthermore, students can gradually understand why some "classic" methods fail when solving DAE problems. Through this comparative analysis method, students' independent critical thinking and evaluation abilities when facing problems can be gradually improved, ensuring that students can more clearly select appropriate methods in the process of numerical selection in the future [2]. In numerical analysis, the core teaching objective of the course is to help students master the basic principles of discretization solutions for continuous mathematical problems, understand the methods, and possess error analysis capabilities. Currently, in the process of learning numerical analysis for many undergraduates and even postgraduates, the main line of the course still focuses on the initial value of ordinary differential equations, which involves multiple different numerical methods, including the stability and convergence analysis of the Runge-Kutta method. Good teaching and analysis of this content can further help students build a sound knowledge system, and also enable students to improve the quality of modeling when dealing with complex modeling and simulation problems in the real world, deepen the teaching content, so that students can better solve problems when applying this content in the future, and make mathematical models truly serve as a tool in various industries in China.

3.2 Construction of an Interconnected and Hierarchical Knowledge Network

To help students better understand the Runge-Kutta method for a class of nonlinear functional differential-algebraic equations, it is necessary to construct a complete knowledge system and comprehensively apply the following knowledge:

Master the basic content of numerical solutions for ordinary differential equations, including the judgment of convergence order and local truncation error;

Analyze the concept of functionals, and understand the delay property existing in the process of data analysis and processing, which leads to the emergence of infinite-dimensional problems;

Analyze the index concept, compatible initial value conditions, and other contents involved in the theory of differential-algebraic equations;

Construct a unified perspective of linear multi-step methods and Runge-Kutta methods, and conduct matrix analysis well, including the stability analysis of matrices and the understanding of a series of data such as matrix norms and eigenvalues.

In the practice of teaching, progressive discussion topics should be designed. For example, starting from linear scalars, analyze delay differential equations and understand the stability of simple R-K methods; on this basis, introduce algebraic constraints, discuss DAE problems of index 1, and fully reveal the impact of constraint processing on the convergence order; at the same time, combine FDAE cases that integrate delays and nonlinearity. This helps students quickly connect previously unfamiliar knowledge points to form a unique knowledge network, analyze, understand, and discuss the internal logical relationships and theories between different concepts, and achieve a leap from "point-like learning" to "structured learning" [3].

3.3 Cultivation of Abstract Thinking and Mathematical Modeling Abilities

Learning the Runge-Kutta method can stimulate students' scientific research interest and innovative awareness. In the process of mathematical knowledge research, this research content is a current research hotspot. Since the content is relatively novel, it is a completely unfamiliar field for many students, and students' curiosity for knowledge will continue to increase during this period. Moreover, the numerical analysis of FDAEs is an active research field with many open problems, such as the effective handling of high-index problems and adaptive compensation strategies. These numerical problems further enhance students' enthusiasm and initiative in learning. The growth of students' curiosity about this content also enables them to more actively solve difficult problems when facing difficulties. In teaching practice, teachers need to set up literature discussion and review sessions, aiming to allow students to independently read simplified reviews or classic papers in this field, so that students can have a clear understanding of the content of the course before learning [4]. Teachers need to encourage all students to point out unsolved problems and difficult points in the papers and put forward their own ideas, whether they are preliminary improvement ideas or in-depth improvement ideas in the future. The ultimate goal is to enable students to achieve effective knowledge transfer, innovation, and exploration, improve teaching quality, and help students build a sound knowledge system and model construction for future computational research. To further ensure the teaching effect and reduce students' fear of difficulties when learning this content.

3.4 Improvement of Teaching Methods

Teachers also need to carefully design teaching methods. They can adopt case-driven methods, visualization tool-assisted hierarchical teaching, and other approaches to improve the quality of education and teaching. When choosing the case-driven teaching method, teachers should always use specific FDAEs with practical backgrounds as examples throughout the theoretical explanation to avoid relatively boring derivation work in mathematics teaching. For example, in practical teaching, transistor models in analog circuits or population dynamics models can be used to stimulate students' thinking about the problem. Visual teaching tools can also be selected for assistance. For instance, in computer-aided teaching, different programs can be written to dynamically show students the solution effects and quality of different solution methods when dealing with the same problem. Under stable and unstable parameters, it is also necessary to consider the behavioral differences of numerical solutions under stable and unstable parameters. When a picture of unstable explosion is presented to students, this method is the most direct and effective, and has stronger persuasiveness than any other written text. For undergraduates, hierarchical teaching should be emphasized, with the focus on understanding concepts, observing phenomena, and clarifying the recognition of their importance. For postgraduates, it is necessary to master various different techniques and improve the ability to study literature. Currently, it is essential to strengthen programming practice. In programming practice teaching, more emphasis is placed on combining theoretical analysis with programming implementation methods. Relevant assignments should be assigned to allow students to independently solve problems, construct a relatively simple diagonally implicit R-K method for effective solution, verify convergence, and observe its stability conditions. Enabling students to independently form a complete closed-loop training of theoretical algorithms and code results can help students deeply understand the teaching content of this course [5].

4. Conclusion

In summary, when analyzing a class of nonlinear functional differential-algebraic equations using the Runge-Kutta method, the regular Runge-Kutta method can yield theoretical results of stability, consistency, and convergence under corresponding conditions, and the relevant proof contents can be obtained. The effectiveness of this method is verified through numerical examples.

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