

Optimal Strategy for Asset-Liability Management under Regime-Switching

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Abstract: To capture the interactive effects of market liquidity regimes, price impacts, and liability levels on investment strategies, a Markov regime-switching mechanism is incorporated into asset-liability management, establishing a stochastic optimal control model with state-dependent liability processes. The results indicate that the liability growth rate influences strategy selection by altering investors' risk tolerance and repayment pressure: investors adopt conservative strategies to hedge against repayment risk under high growth rates, while pursuing aggressive strategies for higher returns under low growth rates. Regime transitions in market liquidity moderate the aggressiveness of strategies, and the joint effect of liabilities and market frictions further influences the adjustment speed and long-term robustness of trading strategies.

Keywords: Markov Regime Switching; Liabilities; Stochastic Optimal Control Model; Hamilton-Jacobi-Bellman (HJB) Equation; Riccati Differential Equation System

1. Introduction

Asset-liability management (ALM) is a core function for financial institutions such as insurance companies, pension funds, and banks. Its primary objective is to achieve financial goals through optimal asset allocation while meeting current and future liability requirements. Traditional ALM models are often based on the assumption of a static market environment, neglecting the impact of macroeconomic factors, policy adjustments, and market sentiment on the dynamic evolution of asset returns, volatility, and liability demands. To more accurately capture these characteristics, this paper introduces a Markov regime-switching mechanism into the ALM research framework.

Chen et al. [1], Xie [2] and Li et al. [3] applied this model to continuous-time ALM problems,

demonstrating that state-dependent asset returns and liability dynamics significantly influence optimal investment strategies. Wei et al. [4] further extended regime-switching to multi-period ALM problems, emphasizing the importance of studying regime shifts for asset allocation and risk management. Recently, Chen et al. [5] explored a robust ALM problem under the CRRA utility framework with regime-switching in a continuous-time setting, effectively addressing decision-making under model uncertainty.

With the continuous development of market microstructure theory, incorporating market frictions into asset-liability management has become increasingly important. Most traditional studies assume an ideal frictionless market, overlooking real-world constraints such as transaction costs, liquidity restrictions, and price impacts. In practice, large trades by institutional investors often lead to significant price distortions [6-9]. Berry-Stöizle [10] first introduced market frictions into the study of ALM for property insurance. Berry-Stöizle [11] further revealed the influence of liquidity risk and transaction costs on asset allocation strategies. However, such research primarily relies on numerical methods and still lacks theoretical support in the form of analytical solutions, especially in the context of multi-asset portfolios and complex friction scenarios.

In recent years, several studies have begun integrating market frictions with regime-switching mechanisms. Ma et al. [12], working within the framework of Markovian jump linear systems, solved for the optimal strategy via coupled Riccati differential equations and found that the type of price impact significantly alters investors' trading behavior patterns. Yan et al. [13] proposed a "target tracking strategy," further revealing the opposing mechanisms through which temporary and permanent price impacts influence trading behavior.

To date, no study has systematically incorporated

liability management, a Markov regime-switching mechanism, and market frictions involving both temporary and permanent price impacts into a unified analytical framework. Therefore, this paper constructs an asset-liability management model that integrates regime-switching and price impacts. The model assumes that the dynamics of assets follow a multi-dimensional geometric Brownian motion, introduces a mean-reverting return prediction signal to capture return predictability in financial markets, and simultaneously models the liability process as a regime-dependent geometric Brownian motion. Based on the dynamic programming principle, this paper transforms the ALM problem into a computational problem involving a system of coupled Riccati differential equations, and demonstrates that the value function and the optimal feedback trading strategy can be expressed through the solution of this system.

2. Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$, which is generated by an n -dimensional standard Brownian motion $(W_t^p)_{t \geq 0}$, an m -dimensional standard Brownian motion $(W_t^f)_{t \geq 0}$, and a continuous-time, homogeneous, stationary Markov chain $(\alpha_t)_{t \geq 0}$, all of which are mutually independent. The Markov chain $(\alpha_t)_{t \geq 0}$ is defined on a finite, canonical state space $\Xi = \{e_1, e_2, \dots, e_k\}$, where each state is represented by a unit vector e_i (with the i -th component being 1 and all others 0). Its transition intensity matrix is given by $\Pi = (\pi_{ij})_{1 \leq i, j \leq k}$ where for each $i \neq j$, $\pi_{ij} \geq 0$, and $\pi_{i,i} = -\sum_{j \neq i} \pi_{ij}$. The matrix element π_{ij} represents the instantaneous transition intensity of the Markov chain $(\alpha_t)_{t \geq 0}$ switching from state i to state j .

We consider a financial market comprising one risk-free asset and n risky assets. The price dynamics of the risky assets are described by a multi-dimensional geometric Brownian motion:

$$dP_t = \text{diag}(P_t) \left[(r_f(\alpha_t) \mathbf{1} + \mathbf{B}(\alpha_t) \mathbf{f}_t) dt + \sigma_p(\alpha_t) dW_t^p \right], \quad (1)$$

where $P_t = (p_{1t}, p_{2t}, \dots, p_{nt})^\top$, $\text{diag}(P_t)$ denotes the diagonal matrix whose diagonal elements are the components of P_t ; W_t^p is an n -dimensional $\mathcal{F}$ -adapted Brownian motion

capturing market randomness; $r_f(\alpha_t)$ represents the state-dependent risk-free rate; and $\mu_{pt}(\alpha_t) := r_f(\alpha_t) \mathbf{1} + \mathbf{B}(\alpha_t) \mathbf{f}_t \in \mathbb{R}^n$ and $\mathbf{f}_t \in \mathbb{R}^{n \times n}$ denote the state-dependent drift vector and volatility coefficient matrix of the risky assets, respectively.

In Equation (1), $\mathbf{f}_t = (f_{1t}, f_{2t}, \dots, f_{nt})^\top \in \mathbb{R}^m$ is a vector of return predictors associated with the state-dependent factor loading matrix $\mathbf{B}(\alpha_t) \in \mathbb{R}^{n \times m}$, whose dimension can differ from the number of risky assets. The return predictor f_t is assumed to follow a stationary Ornstein-Uhlenbeck process:

$$df_t = -\Phi(\alpha_t) f_t dt + \sigma_f(\alpha_t) dW_t^f, \quad (2)$$

Where $\Phi(\alpha_t) \in \mathbb{R}^{m \times m}$ is the state-dependent mean-reversion matrix, W_t^f is an m -dimensional $\mathcal{F}$ -adapted Brownian motion which may be correlated with W_t^p , and $\sigma_f(\alpha_t) \in \mathbb{R}^{m \times m}$ is the state-dependent volatility coefficient matrix of the return predictors. Furthermore, the state-dependent covariance matrices for the risky assets and the return predictors are given by $\Sigma_p(\alpha_t) := \sigma_p(\alpha_t) \sigma_p(\alpha_t)^\top$ and $\Sigma_f(\alpha_t) := \sigma_f(\alpha_t) \sigma_f(\alpha_t)^\top$, respectively.

Let the portfolio's positions in the risky assets be denoted by $x_t = (x_{1t}, x_{2t}, \dots, x_{nt})^\top$. Following Gârleanu and Pedersen [8], the investor dynamically adjusts the portfolio by choosing a trading intensity $(\tau_t)_{t \geq 0}$, which determines the instantaneous rate of change of the positions $(x_t)_{t \geq 0}$ as follows:

$$dx_t = \tau_t dt \quad (3)$$

where $\tau_t \geq 0$ is the time-dependent trading strategy, reflecting the investor's dynamic portfolio adjustment decisions under different market regimes.

The company's asset value at time t is defined as a_t , and it is assumed to satisfy the following stochastic differential equation in a frictionless market:

$$da_t = r_f(\alpha_t) (a_t - \mathbf{x}_t \mathbf{1}) dt + \mathbf{x}_t \text{diag}(P_t)^{-1} dP_t. \quad (4)$$

However, market frictions are prevalent in real-world trading. For an instantaneous trading strategy τ_t , it is assumed that the average execution price increases by $\frac{1}{2} \Lambda(\alpha_t) \tau_t$, where $\Lambda(\alpha_t) \in \mathbb{R}^{n \times n}$ is a state-dependent symmetric positive definite matrix (commonly referred to as the multidimensional Kyle's λ matrix), which measures the level of transaction costs. Let $\bar{P}_t$ denote the average execution price under the

trading intensity τ_t . Then, we have:

$$\begin{aligned}\tilde{\mathbf{P}}_t &= \mathbf{P}_t + \frac{1}{2} \text{diag}(\mathbf{P}_t) \Lambda(\alpha_t) \tau_t \\ &= \text{diag}(\mathbf{P}_t) \left(\mathbf{1} + \frac{1}{2} \Lambda(\alpha_t) \tau_t \right)\end{aligned}\quad (5)$$

Therefore, combining with Equation (5), the temporary transaction cost $\text{TC}(\tau_t)dt$ induced by the trading strategy τ_t is expressed as:

$$\begin{aligned}\text{TC}(\tau_t)dt &= \left(\text{diag}(\mathbf{P}_t)^{-1} \tau_t dt \right) \tilde{\mathbf{P}}_t \\ &\quad - \tau_t \mathbf{1} dt \\ &= \frac{1}{2} \tau_t \Lambda(\alpha_t) \tau_t dt\end{aligned}\quad (6)$$

Furthermore, trading activity simultaneously induces both permanent and temporary price impacts. When an investor submits a buy (or sell) order, the order causes a permanent increase (or decrease) in the price of the risky asset. To capture this, a price distortion process D_t is introduced, whose dynamics satisfy the following equation:

$$dD_t = -R(\alpha_t)D_t dt + C(\alpha_t)\tau_t dt, \quad (7)$$

where $D_0 = 0_{n \times 1}$, $C(\alpha_t) \in R^{n \times n}$ is the state-dependent permanent price impact matrix, describing the permanent price change per unit of trading intensity; and $R(\alpha_t) \in R^{n \times n}$ is the state-dependent market resilience matrix, characterizing the decay rate of the price distortion.

In the presence of both permanent (Equation (7)) and temporary (Equation (6)) price impacts, the dynamics of the firm's asset value in (4) further follow the equation below:

$$\begin{aligned}da_t &= r_f(a_t - x_t^\top \mathbf{1}) dt + x_t^\top [(\text{diag} \mathbf{P}_t)^{-1} \\ &\quad \& d\mathbf{P}_t + dD_t] - \text{TC}(\tau_t)dt \\ &\&= \left\{ r_f(a_t) a_t + x_t^\top [B(\alpha_t) f_t dt - R(\alpha_t) D_t + C(\alpha_t) \tau_t] \right. \\ &\quad \& \left. \& D_t + C(\alpha_t) \tau_t - \frac{1}{2} \tau_t^\top \Lambda(\alpha_t) \tau_t \right\} dt \\ &\quad \& + x_t^\top \sigma_P(\alpha_t) dW_t^P,\end{aligned}\quad (8)$$

In this equation, $r_f(\alpha_t)a_t$ captures the risk-free return, while $\mathbf{x}_t = [B(\alpha_t) f_t dt - R(\alpha_t) D_t + C(\alpha_t) \tau_t]$ represents the return from risky assets incorporating price distortion. The term $x_t^\top \sigma_P(\alpha_t) dW_t^P$ reflects changes in asset value due to market volatility, and $-\frac{1}{2} \tau_t^\top \Lambda(\alpha_t) \tau_t$ accounts for the temporary transaction costs and the direct capital outflow resulting from trade execution. Following Xie [2], Zhang and Chen [14], as well as Zhu et al. [15], this paper models the

company's liability process using a geometric Brownian motion. Furthermore, it is assumed that the liability process is influenced by a Markov regime-switching mechanism, specified as follows:

$$dl_t = l_t [\mu_l(\alpha_t) dt + \sigma_l(\alpha_t) dW_t^l] \quad (9)$$

where W_t^l is an $\mathcal{F}$ -adapted standard Brownian motion, which may be correlated with W_t^P and W_t^f . The correlation coefficient vectors are denoted as $\rho_{pl} \in R^n$ and $\rho_{fl} \in R^n$, respectively. To simplify notation, we define:

$$\begin{aligned}\Sigma_l(\alpha_t) &:= \sigma_l(\alpha_t)^2 \in R, \quad \Sigma_{pl}(\alpha_t) := \sigma_p(\alpha_t) \rho_{pl} \\ \sigma_l(\alpha_t) &\in R^n \text{ and } \Sigma_{fl}(\alpha_t) := \sigma_f(\alpha_t) \rho_{fl} \sigma_l(\alpha_t) \in R^m.\end{aligned}$$

Then, combining Equation (8) and Equation (9), the company's net asset value $S_t := a_t - l_t$ satisfies:

$$\begin{aligned}dS_t &= \left\{ r_f(\alpha_t) S_t + (r_f(\alpha_t) - \mu_l(\alpha_t)) l_t \right. \\ &\quad \& \left. + \mathbf{x}_t [B(\alpha_t) f_t - R(\alpha_t) D_t \right. \\ &\quad \& \left. + C(\alpha_t) \tau_t] - \frac{1}{2} \tau_t^\top \Lambda(\alpha_t) \tau_t \right\} dt \\ &\quad \& + \mathbf{x}_t^\top \sigma_P(\alpha_t) dW_t^P - \sigma_l(\alpha_t) l_t dW_t^l,\end{aligned}\quad (10)$$

3. Objective Function

The investor's objective is to maximize the discounted sum of the net asset value returns, based on a local mean-variance criterion, over the trading horizon $t \in [0, T]$. Accordingly, we consider the following objective function:

$$\begin{aligned}J(t, x, D, f, l, e_i; \tau) &:= E_t \left[\int_t^T e^{-\rho(s-t)} \left(E_s [dS_s - r_f(\alpha_s) S_s ds] \right. \right. \\ &\quad \& \left. \left. - \frac{\gamma}{2} \text{Var}_s [dS_s - r_f(\alpha_s) S_s ds] \right) \right] \\ &\&= E_t \left[\int_t^T e^{-\rho(s-t)} \left\{ (r_f(\alpha_s) - \mu_l(\alpha_s)) l_s \right. \right. \\ &\quad \& \left. \left. + x_s^\top [B(\alpha_s) f_s - R(\alpha_s) D_s + C(\alpha_s) \tau_s] \right. \right. \\ &\quad \& \left. \left. - \frac{1}{2} \tau_s^\top \Lambda(\alpha_s) \tau_s - \frac{\gamma}{2} [x_s^\top \Sigma_P(\alpha_s) x_s \right. \right. \\ &\quad \& \left. \left. + \Sigma_l(\alpha_s) l_s^2 - 2x_s^\top \Sigma_P(\alpha_s) l_s] \right\} ds \right],\end{aligned}\quad (11)$$

and seek an optimal trading strategy $\tau^* \in \mathcal{A}$ that maximizes the objective function, i.e.,

$$\begin{aligned}J(t, x, D, f, l, e_i; \tau^*) &:= \max_{\tau \in \mathcal{A}} J(t, x, D, f, l, e_i; \tau)\end{aligned}\quad (12)$$

subject to the following system constraints:

$$\begin{aligned}dx_t &= \tau_t dt, \\ df_t &= -\Phi(\alpha_t) f_t dt + \sigma_f(\alpha_t) dW_t^f, \\ dD_t &= -R(\alpha_t) D_t dt + C(\alpha_t) \tau_t dt, \\ dl_t &= l_t [\mu_l(\alpha_t) dt + \sigma_l(\alpha_t) dW_t^l],\end{aligned}\quad (13)$$

Here, $(\alpha_t)_{t \geq 0}$ is a continuous-time Markov chain with transition intensity matrix Π , $\rho > 0$ is the discount rate, $\gamma > 0$ is the risk aversion parameter, and $E_t[\cdot]$ and $\text{Var}_t[\cdot]$ denote the conditional

expectation and conditional variance operators, respectively, given the information set $\mathcal{F}_t$ at time t .

Note: To simplify notation, the objective function (11) is denoted as $J(t, x, D, f, l, i; \tau^*) := J(t, x, D, f, l, e_i; \tau^*)$, and the value function is denoted as $V(t, x, D, f, l, i) := V(t, x, D, f, l, e_i)$. Model parameters under state i are uniformly represented as: $B_i := B(e_i)$, $r_i := r(e_i)$, $C_i := C(e_i)$, $\Lambda_i := \Lambda(e_i)$, $\Sigma_i^f := \Sigma_f(e_i)$, $\Sigma_i^p := \Sigma_p(e_i)$, $\Sigma_i^l := \Sigma_l(e_i)$, $\Phi_i := \Phi(e_i)$, $R_i := R(e_i)$, $\mu_i^l := \mu_l(e_i)$, $\Sigma_i^f := \Sigma_f(e_i)$, $\Sigma_i^p := \Sigma_p(e_i)$, $\Sigma_i^l := \Sigma_l(e_i)$. According to dynamic programming theory, the Hamilton-Jacobi-Bellman (HJB) equation corresponding to the value function V is given by:

$$\begin{aligned} & \rho V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i) \\ &= \max_{\tau \in \mathbb{R}^+} \left\{ \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial t} \right. \\ &+ L^\tau V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i) + (r_i - \mu_i^l) l \\ &+ \mathbf{x}^\top [\mathbf{B}_i \mathbf{f} - \mathbf{R}_i \mathbf{D} + \mathbf{C}_i \tau] - \frac{1}{2} \tau^\top \Lambda_i \tau \\ &- \frac{\gamma}{2} (\mathbf{x}^\top \Sigma_i^p \mathbf{x} + \Sigma_i^l l^2 - 2 \mathbf{x}^\top \Sigma_i^p l) \\ &+ \sum_{j \neq i} \pi_{i,j} [V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, j) \\ &- V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)] \} \end{aligned} \quad (14)$$

where,

$$\begin{aligned} & L^\tau V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i) \\ &= \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{x}} \tau - \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{f}} \Phi_i \mathbf{f} \\ &+ \frac{1}{2} \text{tr} \left(\frac{\partial^2 V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{f} \partial \mathbf{f}} \Sigma_i^f \right) + \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial l} \\ &\mu_i^l l + \frac{1}{2} \frac{\partial^2 V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial l^2} \Sigma_i^l l^2 + \frac{\partial^2 V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{f} \partial l} \\ &\Sigma_i^p l + \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{D}} (-\mathbf{R}_i \mathbf{D} + \mathbf{C}_i \tau), \end{aligned} \quad (15)$$

the terminal condition is $V(T, x, D, f, l, i) = 0$.

4. Study of the Optimal Trading Strategy

Following Ma G et al. [12], the solution to the aforementioned HJB equation (14) can be obtained by solving a system of coupled Riccati equations. Denote the time derivative of a function $F(t)$ as $\dot{F}(t)$.

Theorem 1 The solution to the HJB equation (14) admits the following quadratic form:

$$\begin{aligned} V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i) &= A_c(t, i) + A_l(t, i) \\ &+ \frac{1}{2} \begin{pmatrix} \mathbf{x} \\ \mathbf{D} \\ \mathbf{f} \\ l \end{pmatrix}^\top \mathbf{Q} \begin{pmatrix} \mathbf{x} \\ \mathbf{D} \\ \mathbf{f} \\ l \end{pmatrix} \end{aligned} \quad (16)$$

Where,

$$\mathbf{Q} = \begin{pmatrix} -A_{xx}(t, i) & A_{xD}(t, i) & A_{xf}(t, i) & A_{xl}(t, i) \\ A_{xD}^\top(t, i) & A_{DD}(t, i) & A_{Df}(t, i) & A_{Dl}(t, i) \\ A_{xf}^\top(t, i) & A_{Df}^\top(t, i) & A_{ff}(t, i) & A_{fl}(t, i) \\ A_{xl}^\top(t, i) & A_{Dl}^\top(t, i) & A_{fl}^\top(t, i) & A_{ll}(t, i) \end{pmatrix},$$

and

$A_{xx}(t, i), A_{ff}(t, i), A_{ll}(t, i), A_{DD}(t, i), A_{xf}(t, i), A_{xD}(t, i), A_{Df}(t, i), A_{xl}(t, i), A_{Dl}(t, i), A_{fl}(t, i), A_l(t, i), A_c(t, i)$ are given by the solution to the following system of coupled Riccati differential equations:

$$\begin{aligned} & \dot{A}_{xx}(t, i) = \rho A_{xx}(t, i) - \gamma \Sigma_i^p \\ & \& - \sum_{j=1}^m \pi_{ij} A_{xx}(t, j) + E_x(t, i)^\top \Lambda_i^{-1} E_x(t, i), \\ & \dot{A}_{xD}(t, i) = \rho A_{xD}(t, i) + r_i 1_n + R_i \\ & \& + A_{xD}(t, i) R_i - \sum_{j=1}^m \pi_{ij} A_{xD}(t, j) \\ & \& - E_x(t, i)^\top \Lambda_i^{-1} E_D(t, i), \\ & \dot{A}_{DD}(t, i) = \rho A_{DD}(t, i) + R_i^\top A_{DD}(t, i) \\ & \& + A_{DD}(t, i) R_i - \sum_{j=1}^m \pi_{ij} A_{DD}(t, j) \\ & \& - E_D(t, i)^\top \Lambda_i^{-1} E_D(t, i), \\ & \dot{A}_{ff}(t, i) = \rho A_{ff}(t, i) + \Phi_i^\top A_{ff}(t, i) \\ & \& + A_{ff}(t, i)^\top \Phi_i - \sum_{j=1}^m \pi_{ij} A_{ff}(t, j) \\ & \& - E_f(t, i)^\top \Lambda_i^{-1} E_f(t, i), \\ & \dot{A}_{xf}(t, i) = \rho A_{xf}(t, i) + A_{xf}(t, i) \Phi_i \\ & \& - B_i - \sum_{j=1}^m \pi_{ij} A_{xf}(t, j) \\ & \& - E_x(t, i)^\top \Lambda_i^{-1} E_f(t, i), \\ & \dot{A}_{Df}(t, i) = \rho A_{Df}(t, i) + R_i^\top A_{Df}(t, i) \\ & \& + A_{Df}(t, i) \Phi_i - \sum_{j=1}^m \pi_{ij} A_{Df}(t, j) \\ & \& - E_D(t, i)^\top \Lambda_i^{-1} E_f(t, i), \\ & \dot{A}_{ll}(t, i) = \rho A_{ll}(t, i) + \gamma \Sigma_i^l - 2 \mu_i^l A_{ll}(t, i) \\ & \& - A_{ll}(t, i) \Sigma_i^l - \sum_{j=1}^m \pi_{ij} A_{ll}(t, j) \\ & \& - E_l(t, i)^\top \Lambda_i^{-1} E_l(t, i), \\ & \dot{A}_{xl}(t, i) = \rho A_{xl}(t, i) - A_{xl}(t, i) \mu_i^l \\ & \& - \gamma \Sigma_i^p - \sum_{j=1}^m \pi_{ij} A_{xl}(t, j) \\ & \& - E_x(t, i)^\top \Lambda_i^{-1} E_l(t, i), \\ & \dot{A}_{Dl}(t, i) = \rho A_{Dl}(t, i) - A_{Dl}(t, i) \mu_i^l \\ & \& + R_i^\top A_{Dl}(t, i) - \sum_{j=1}^m \pi_{ij} A_{Dl}(t, j) \\ & \& - E_D(t, i)^\top \Lambda_i^{-1} E_l(t, i), \\ & \dot{A}_{fl}(t, i) = \rho A_{fl}(t, i) + A_{fl}(t, i) \Phi_i \\ & \& - A_{fl}(t, i) \mu_i^l - \sum_{j=1}^m \pi_{ij} A_{fl}(t, j) \\ & \& - E_f(t, i)^\top \Lambda_i^{-1} E_l(t, i), \\ & \dot{A}_l(t, i) = \rho A_l(t, i) - r_i 1_n + \mu_i^l 1_n \\ & \& - A_l(t, i) \mu_i^l - A_{fl}(t, i)^\top \Sigma_i^f \\ & \& - \sum_{j=1}^m \pi_{ij} A_l(t, j), \\ & \dot{A}_c(t, i) = \rho A_c(t, i) - \frac{1}{2} \text{tr}(A_{ff}(t, i) \Sigma_i^f) \\ & \& - \sum_{j=1}^m \pi_{ij} A_c(t, j). \end{aligned} \quad (17)$$

where

$$\begin{cases} E_x(t,i) := C_i^\top - A_{xx}(t,i) \\ \quad + C_i^\top A_{xD}(t,i)^\top, \\ E_D(t,i) := A_{xD}(t,i) + C_i^\top A_{DD}(t,i), \\ E_f(t,i) := A_{xf}(t,i) + C_i^\top A_{Df}(t,i), \\ E_l(t,i) := A_{xl}(t,i) + C_i^\top A_{Dl}(t,i). \end{cases} \quad (18)$$

The terminal condition for system (15) is $A_{xx}(T,i) = 0_{n \times n}$, $A_{DD}(T,i) = 0_{n \times n}$, $A_{xD}(T,i) = 0_{n \times n}$, $A_{ff}(T,i) = 0_{m \times m}$, $A_{xf}(T,i) = 0_{n \times m}$, $A_{Df}(T,i) = 0_{n \times m}$, $A_{xl}(T,i) = 0_{n \times 1}$, $A_{Dl}(T,i) = 0_{n \times 1}$, $A_{fl}(T,i) = 0_{m \times 1}$, $A_{ll}(T,i) = A_l(T,i) = A_c(T,i) = 0$.

Accordingly, the optimal trading strategy can be derived as:

$$\begin{aligned} \tau^*(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i) \\ = \Lambda_i^{-1} [\mathbf{E}_x(t,i) \mathbf{x} + \mathbf{E}_D(t,i) \mathbf{D} \\ + \mathbf{E}_f(t,i) \mathbf{f} + \mathbf{E}_l(t,i) l] \end{aligned} \quad (19)$$

Proof For $i=1, \dots, k$, taking the first-order condition with respect to τ in the HJB equation (14)

yields $\tau = \Lambda_i^{-1} [C_i^\top x + \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{x}} + C_i^\top \frac{\partial V(t, \mathbf{x}, \mathbf{D}, \mathbf{f}, l, i)}{\partial \mathbf{D}}]$. We now postulate a solution for V of the form given in (16), where the symmetric matrices $A_{xx}(t,i)$, $A_{ff}(t,i)$, $A_{DD}(t,i)$; the matrices $A_{xf}(t,i)$, $A_{xD}(t,i)$, $A_{Df}(t,i)$, $A_{xl}(t,i)$, $A_{Dl}(t,i)$, $A_{fl}(t,i)$; and the scalar functions $A_{ll}(t,i)$, $A_l(t,i)$, A_c are all state-dependent parameters to be determined. Substituting this ansatz into the first-order condition and then back into the HJB equation (14), we derive the system of coupled Riccati equations (17) and the optimal trading strategy τ^* (19) by matching the coefficients of the terms $x^\top (\cdot) x$, $x^\top (\cdot) f$, $D^\top (\cdot) D$, $D^\top (\cdot) l$, $x^\top (\cdot) D$, $x^\top (\cdot) l$, $D^\top (\cdot) f$, $D^\top (\cdot) l$, $f^\top (\cdot) f$, and the constant term.

This theorem shows that, within a framework accounting for market liquidity, the optimal investment decision problem can be transformed into solving a system of coupled Riccati equations. By solving system (17) and determining the analytical forms of the coefficient matrices, we obtain the explicit optimal trading strategy (19). This strategy exhibits a clear linear feedback form, where the trading intensity is determined by the current holdings x , the price distortion process D , the return prediction signal f , the liability level l ,

and the liquidity state, with dynamic adjustment achieved through the time-varying coefficient matrices E_x, E_D, E_f, E_l .

When $C_i = R_i = 0_{n \times n}$ ($i=1, \dots, k$), the HJB equation (14) simplifies significantly. Under this condition, market frictions arise only from linear temporary price impacts. From the dynamics in equation (7), if $C_i = R_i = 0_{n \times n}$ for all i , then $D_i = 0_{n \times 1}$. The following corollary provides a classical analytical solution to the HJB equation (14) under this specific condition.

Corollary 1 When $C_i = R_i = 0_{n \times n}$, the solution to the HJB equation (14) is given by:

$$\begin{aligned} V^{np}(t, \mathbf{x}, \mathbf{f}, l, i) \\ = -\frac{1}{2} \mathbf{x}^\top \mathbf{A}_{xx}^{np}(t,i) \mathbf{x} + \frac{1}{2} \mathbf{f}^\top \mathbf{A}_{ff}^{np}(t,i) \mathbf{f} \\ + \frac{1}{2} A_{ll}^{np}(t,i) l^2 + \mathbf{x}^\top \mathbf{A}_{xl}^{np}(t,i) \mathbf{f} + \mathbf{x}^\top \mathbf{A}_{xl}^{np}(t,i) l \\ + \mathbf{f}^\top \mathbf{A}_{fl}^{np}(t,i) l + A_c^{np}(t,i) l, \end{aligned} \quad (20)$$

where

$A_{xx}^{np}(t,i)$, $A_{ff}^{np}(t,i)$, $A_{ll}^{np}(t,i)$, $A_{xf}^{np}(t,i)$, $A_{xl}^{np}(t,i)$, $A_{fl}^{np}(t,i)$, $A_l^{np}(t,i)$, $A_c^{np}(t,i)$ is the solution to the following system of coupled equations:

$$\begin{cases} \dot{A}_{fl}^{np}(t,i) = \rho A_{fl}^{np}(t,i) + \Phi_i^\top A_{fl}^{np}(t,i) \\ \quad \& - A_{fl}^{np}(t,i) \mu_i^l - \sum_{j=1}^m \pi_{ij} A_{fl}^{np}(t,j) \\ \quad \& - A_{xf}^{np}(t,i)^\top \Lambda_i^{-1} A_{xl}^{np}(t,i), \\ \dot{A}_l^{np}(t,i) = \rho A_l^{np}(t,i) - r_l 1_n + \mu_i^l 1_n \\ \quad \& - \mu_i^l A_l^{np}(t,i) - A_{fl}^{np}(t,i)^\top \Sigma_i^l \\ \quad \& - \sum_{j=1}^m \pi_{ij} A_l^{np}(t,j), \\ \dot{A}_c^{np}(t,i) = \rho A_c^{np}(t,i) - \frac{1}{2} \text{tr}(A_{ff}^{np}(t,i) \Sigma_i^f) \\ \quad \& - \sum_{j=1}^m \pi_{ij} A_c^{np}(t,j), \\ \dot{A}_{xx}^{np}(t,i) = \rho A_{xx}^{np}(t,i) - \gamma \Sigma_i^{pp} \\ \quad \& - \sum_{j=1}^m \pi_{ij} A_{xx}^{np}(t,j) + A_{xx}^{np}(t,i)^\top \Lambda_i^{-1} A_{xx}^{np}(t,i), \\ \dot{A}_{ff}^{np}(t,i) = \rho A_{ff}^{np}(t,i) + \Phi_i^\top A_{ff}^{np}(t,i) \\ \quad \& + A_{ff}^{np}(t,i)^\top \Phi_i - \sum_{j=1}^m \pi_{ij} A_{ff}^{np}(t,j) \\ \quad \& - A_{xf}^{np}(t,i)^\top \Lambda_i^{-1} A_{xf}(t,i), \\ \dot{A}_{xf}^{np}(t,i) = \rho A_{xf}^{np}(t,i) + A_{xf}^{np}(t,i) \Phi_i - B_i \\ \quad \& - \sum_{j=1}^m \pi_{ij} A_{xf}^{np}(t,j) + A_{xx}^{np}(t,i)^\top \Lambda_i^{-1} A_{xf}^{np}(t,i), \\ \dot{A}_{ll}^{np}(t,i) = \rho A_{ll}^{np}(t,i) + \gamma \Sigma_i^l - 2\mu_i^l A_{ll}^{np}(t,i) \\ \quad \& - A_{ll}^{np}(t,i) \Sigma_i^l - \sum_{j=1}^m \pi_{ij} A_{ll}^{np}(t,j) \\ \quad \& - A_{xl}^{np}(t,i)^\top \Lambda_i^{-1} A_{xl}^{np}(t,i), \\ \dot{A}_{xl}^{np}(t,i) = \rho A_{xl}^{np}(t,i) - A_{xl}^{np}(t,i) \mu_i^l \\ \quad \& - \gamma \Sigma_i^{pl} - \sum_{j=1}^m \pi_{ij} A_{xl}^{np}(t,j) \\ \quad \& + A_{xx}^{np}(t,i)^\top \Lambda_i^{-1} A_{xl}^{np}(t,i), \end{cases} \quad (21)$$

the terminal condition is $A_{xx}^{np}(T,i) = 0_{n \times n}$, $A_{ff}^{np}(T,i) = 0_{m \times m}$, $A_{xf}^{np}(T,i) = 0_{n \times m}$, $A_{xl}^{np}(T,i) = 0_{n \times 1}$, $A_{fl}^{np}(T,i) = 0_{m \times 1}$,

$$A_{il}^{np}(T, i) = A_l^{np}(T, i) = A_c^{np}(T, i) = 0.$$

Accordingly, the optimal trading strategy can be derived as:

$$\begin{aligned} \tau^{np}(t, \mathbf{x}, \mathbf{f}, l, i) \\ = \Lambda_i^{-1} \left[-\mathbf{A}_{xx}^{np}(t, i) \mathbf{x} \right. \\ \left. + \mathbf{A}_{xf}^{np}(t, i) \mathbf{f} + \mathbf{A}_{xl}^{np}(t, i) l \right]. \end{aligned} \quad (22)$$

We conclude this section by presenting a verification theorem.

Theorem 2 Suppose the coupled Riccati system (17) (or (21)) admits a classical solution. Then the value function V in (16) (or V^{np} in (20)) is the solution to the objective function (12), and the investor's optimal trading strategy given in (19) (or (22)) solves the optimization problem (12).

Proof Since the value function in (17) is a quadratic function of the state variables (x, f, D, l) , we follow the standard proof of the verification theorem [16] to show that under the optimal trading strategy τ^* , the processes $(x_t)_{t \geq 0}$ in (3) and $(D_t)_{t \geq 0}$ in (7) admit strong solutions. Specifically, we need to verify that the optimal trading strategy $\tau^* \in A$.

The process $(\alpha_t)_{t \geq 0}$ is a finite-state Markov regime-switching process, independent of the Brownian motion $(W_t^f)_{t \geq 0}$. From equation (2), for any $p \geq 1$, the process f_t belongs to the space $L_{\mathcal{F}}^p(0, T; R^m)$, i.e., f_t is an $\mathcal{F}_t$ -adapted process taking values in R^m and satisfies $E \left[\int_0^T \|f_t\|^p dt \right] < \infty$ which indicates that the expectation of the p -th moment of f_t integrated over $[0, T]$ is finite.

As for the liability process l_t , the coefficients μ_t^l and σ_t^l are bounded and adapted, and independent of the Brownian motion $(W_t^l)_{t \geq 0}$. From equation (9), for any $p \geq 1$, the process $(l_t)_{t \geq 0}$ belongs to the space $L_{\mathcal{F}}^p(0, T; R^n)$. That is, l_t is an $\mathcal{F}_t$ -adapted process taking values in R^n , satisfying $E \left[\int_0^T \|l_t\|^p dt \right] < \infty$.

Next, we substitute the expression for τ^* from (19) into equations (3) and (7), respectively. The resulting dynamics of the variables $(x_t)_{t \geq 0}$ and $(D_t)_{t \geq 0}$ under the optimal trading strategy are given by

$$\begin{aligned} dx_t^* &= \left[\Lambda_i^{-1} (C_i - A_{xx}(t, i) + C_i^\top A_{xD}(t, i)) x_t^* + \Lambda_i^{-1} (A_{xD}(t, i) \right. \\ &\quad \left. + C_i^\top A_{DD}(t, i)) D_t^* + \Lambda_i^{-1} (A_{xf}(t, i) + C_i^\top A_{Df}(t, i)) f_t \right. \\ &\quad \left. + \Lambda_i^{-1} (A_{xl}(t, i) + C_i^\top A_{Dl}(t, i)) l_t \right] dt + dD_t^* = \\ &= \left\{ C_i \Lambda_i^{-1} (C_i - A_{xx}(t, i) + C_i \Lambda_i^{-1} A_{xD}(t, i)) x_t^* + \left[C_i \Lambda_i^{-1} \right. \right. \end{aligned}$$

$$\begin{aligned} &\left. (A_{xD}(t, i) + C_i^\top A_{DD}(t, i)) - R_i \right] D_t^* + C_i \Lambda_i^{-1} (A_{xf}(t, i) \\ &+ C_i \Lambda_i^{-1} A_{Df}(t, i)) f_t + C_i \Lambda_i^{-1} (A_{xl}(t, i) + C_i \Lambda_i^{-1} A_{Dl}(t, i)) \\ &\left. l_t \right\} dt. \end{aligned}$$

We now combine x_t^* and D_t^* into a joint

vector: $Y_t^* = \begin{pmatrix} x_t^* \\ D_t^* \end{pmatrix}$. Define the system matrix

and vectors as follows:

$$\begin{aligned} J(t, i) &= \begin{pmatrix} \Lambda_i^{-1} E_x(t, i) & \Lambda_i^{-1} E_x(t, i) \\ C_i \Lambda_i^{-1} E_D(t, i) & C_i \Lambda_i^{-1} E_D(t, i) - R_i \end{pmatrix}, j(t, i) = \begin{pmatrix} \Lambda_i^{-1} E_f(t, i) \\ C_i \Lambda_i^{-1} E_f(t, i) \end{pmatrix} \\ g(t, i) &= \begin{pmatrix} \Lambda_i^{-1} E_l(t, i) \\ C_i \Lambda_i^{-1} E_l(t, i) \end{pmatrix}, \end{aligned}$$

Then the entire coupled system becomes:

$$dY_t^* = \left[J(t, i) Y_t^* + j(t, i) f_t + g(t, i) l_t \right] dt, Y_0^* = \begin{pmatrix} x_0 \\ 0_{n \times 1} \end{pmatrix}, T$$

his constitutes a linear ordinary differential equation with stochastic driving terms. Since the system matrix, the process f_t , and l_t are all adapted and L^p -integrable, it follows from the existence and uniqueness theory for ordinary differential equations that this system admits a unique strong solution.

5. Numerical Simulations

In this section on numerical simulations, following Ma et al. [12], we assume the model satisfies: $\Lambda_i = \lambda_i \Sigma_i^p$, $C_i = C_i \Sigma_i^p$, $R_i = R_i (\Sigma_i^p)^{-1}$, where for $i=1, \dots, k$, $\lambda_i, C_i, R_i > 0$. We consider two risky assets, i.e., $n = m = k = 2$.

Let e_l and e_h denote the low (l) and high (h) liquidity states, respectively. The low and high liquidity states are characterized by parameters satisfying $\lambda_l \geq \lambda_h, C_l \geq C_h, R_l \leq R_h$. This setup aims to analyze the impact of price impacts and liability levels on the investor's trading strategy under the combined influence of liabilities and regime switching.

Referring to the parameter settings in Ma et al. [12], the baseline parameters are configured as follows: $T=1$, $\gamma=0.4$, $\rho=0.08$, $r_l=r_h=0.01$, $x_0=(0,0)^\top$, $f_0=(1,1)^\top$, $l_0=(1,1)^\top$, $B_l=B_h=I_2$, $\Phi_l=\Phi_h=I_2$, $\Sigma_l^f=0_{2 \times 2}$, $\Sigma_l^l=0.01$, $\Sigma_l^{pl}=\begin{pmatrix} 0.015 & 0 \\ 0 & 0.015 \end{pmatrix}$, $\Sigma_l^p=\Sigma_h^p=\begin{pmatrix} 0.04 & 0.01 \\ 0.01 & 0.04 \end{pmatrix}$ and $\Pi=\begin{pmatrix} -0.05 & 0.05 \\ 0.1 & -0.1 \end{pmatrix}$, the model parameters $\lambda_l, \lambda_h, C_l, C_h, R_l, R_h, \mu_l, \mu_h$ will be specified in subsequent analyses to investigate the impact of different liquidity parameters on the investor's optimal trading strategy.

5.1 Price Impact Effects Under Different Liquidity Regimes

In this section, we first analyze the effect of temporary price impacts (λ_l, λ_h), followed by a discussion of the effects of permanent price impacts (C_l, C_h) and market resilience rates (R_l, R_h). To facilitate comparison with Ma et al. [12], the model parameters $\lambda_l, \lambda_h, C_l, C_h, R_l, R_h$ are set consistent with their study.

5.1.1 Temporary price impact

Setting the parameters as $\lambda_l=1, \lambda_h=0.1, C_l=C_h=4 \times 10^{-3}, R_l=R_h=0.01, \mu_l=\mu_h=0.04$, Figure 1 displays the optimal trading strategy τ^* for Asset 1 and the corresponding sample path of the Markov chain over time.

For comparison, we also present the trading strategies for Asset 1 under the hypothetical scenarios of remaining permanently in the low-liquidity regime (τ_l) and the high-liquidity regime (τ_h) throughout the entire trading horizon. These serve as benchmarks for assessing whether the trading strategy under regime switching is aggressive or conservative relative to the single-regime cases.

The results in Figure 1 show that, under temporary price impact, the investor trades Asset 1 more slowly in the low-liquidity state compared to the permanent low-liquidity scenario ($\tau^* < \tau_l$), indicating relatively conservative behavior. Conversely, in the high-liquidity state, the investor trades faster than in the permanent high-liquidity scenario ($\tau^* > \tau_h$), showing more aggressive trading. This behavior is directly related to market friction costs: low liquidity corresponds to higher transaction cost matrices, which dampen the willingness to adjust positions. The introduction of liabilities alters the risk characteristics of the trading strategy. In the low-liquidity state, the growth of liabilities exacerbates the volatility of net assets, compelling the investor to increase trading speed to dynamically hedge liability risk. In the high-liquidity state, although transaction costs are lower, the presence of liabilities prompts the investor to further scale up trading, actively adjusting positions to address the long-term solvency pressure brought by liabilities. Compared to the liability-free study in Ma et al. [12], the dual effects of regime switching and liabilities in our model lead to a smoother evolution of the trading speed over time. This result indicates that the introduction of liabilities

can mitigate the investor's overreaction to short-term market fluctuations, thereby steering the trading strategy toward long-term stability.

5.1.2 Permanent price impact

The magnitude of the permanent price impact is jointly determined by the permanent price impact coefficients (C_l, C_h) and the market resilience rates (R_l, R_h). Below, we examine the influence of these two factors on the optimal trading strategy separately.

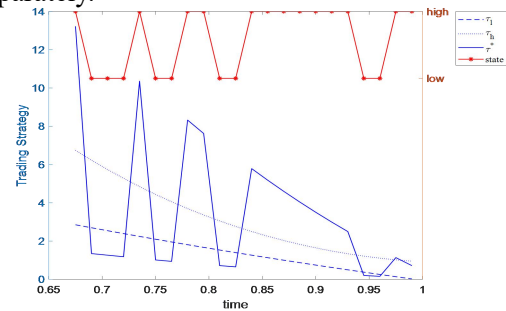


Figure 1. Optimal Trading Strategy τ^* (State Transitions Determined by Temporary Price Impact)

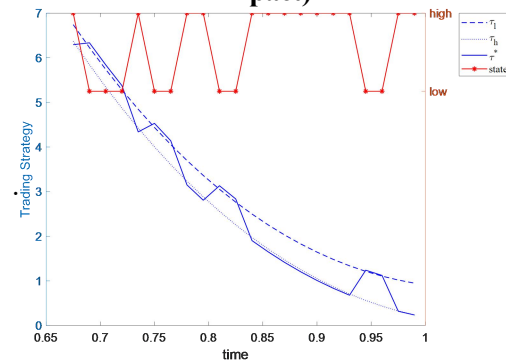


Figure 2. Optimal Trading Strategy τ^* (State Transitions Determined by Permanent Price Impact)

For Figure 2, the parameters are set as $\lambda_l=\lambda_h=0.1, C_l=4 \times 10^{-3}, C_h=1 \times 10^{-3}, R_l=R_h=0.01, \mu_l=\mu_h=0.04$. It can be observed that the investor's trading behavior is more aggressive in the low-liquidity state compared to the high-liquidity state. This occurs because, in a low-liquidity environment, large trades induce significant permanent price impacts. The investor can exploit this by trading actively to create long-term price trends (e.g., prices permanently increase after buying), thereby achieving higher expected returns. This incentive to "influence prices through trading" partially offsets the increased transaction costs associated with low liquidity, encouraging the adoption of a more aggressive trading strategy.

Next, we consider the impact of the market resilience rate on the investor's trading strategy.

Under the parameter settings $\lambda_l = \lambda_h = 0.1$, $C_l = C_h = 4 \times 10^{-3}$, $R_l = 0.01$, $R_h = 0.06$, $\mu_l = \mu_h = 0.04$, Figure 3 shows that the investor adopts a more aggressive trading strategy in the low-liquidity state. Due to the lower market resilience rate (R_l), the permanent price impact caused by the investor's trading activity persists in the market for a longer duration, thereby enhancing the sustainability of the expected investment returns. Conversely, the higher resilience rate (R_h) causes the price impact to dissipate quickly, weakening the long-term return expectations from trading. Consequently, the investor tends to adopt a more short-term and cautious trading approach to avoid potential losses that could arise from price reversals.

Compared to the study of liability-free trading strategies in Ma et al. [12], the presence of liabilities prompts investors to pay greater attention to risk management across different time horizons. The short-term effects of permanent price impacts are diluted by the long-term solvency pressure imposed by liabilities; consequently, their influence on the slope of the trading strategy (i.e., the rate of change of the strategy over time) is relatively minor. Within the Markov regime-switching framework, investors should anticipate the possibility of transitions from low-liquidity to high-liquidity states. This uncertainty drives investors to undertake more proactive, preventive adjustments in the current state based on expectations of future regimes.

5.2 Impact of Liability Growth Rate

To highlight the independent effect of liability dynamics on the trading strategy, we analyze the following two market liquidity scenarios separately. The first scenario assumes a constant market liquidity state, where the parameters determining market liquidity, remain fixed (whether at a high or low level, their values are constant). In this case, only the liability growth rate is influenced by Markov regime switching. The second scenario allows for changes in the market liquidity state, meaning the parameters, take different values in the high and low states. Here, the liability growth rate is also affected by Markov regime switching but cannot, in turn, alter the inherent market liquidity level.

It is important to note that, in both scenarios, the liability growth rate influences trading decisions solely by altering the investor's solvency pressure. The market liquidity level is determined by market-specific parameters such as λ, C, R , and is

independent of the liability growth rate.

When the parameters are set as $\lambda_l = \lambda_h = 0.1$, $C_l = C_h = 4 \times 10^{-3}$, $R_l = R_h = 0.01$, $\mu_l = 1$, $\mu_h = 0.04$, market liquidity remains constant. In this case, the dynamic evolution of the optimal trading strategy in Figure 4 is entirely driven by the liability growth rate. Under a low liability growth rate, investors face lighter long-term solvency pressure, leading to significantly higher risk tolerance. The strategy curve exhibits a steep upward trend, characterized by high-frequency position adjustments and active exploitation of market opportunities. In contrast, under a high liability growth rate, persistent solvency pressure forces investors to prioritize risk control, substantially reducing trading frequency and displaying a clearly conservative tendency.

When the market is in different liquidity states, with parameters set as $\lambda_l = 1$, $\lambda_h = 0.1$, $C_l = 4 \times 10^{-3}$, $C_h = 1 \times 10^{-3}$, $R_l = 0.01$, $R_h = 0.06$, $\mu_l = 1$, $\mu_h = 0.04$, Figure 5 illustrates the combined impact of the liability growth rate and the liquidity state on the trading strategy. On one hand, the liquidity state determines the baseline trading tendency. In the low-liquidity state, where market frictions and transaction costs are high, the investor's trading strategy is relatively conservative ($\tau^* < \tau_l$). Conversely, in the high-liquidity state, where market transaction costs are low, trading is more aggressive ($\tau^* > \tau_h$). On the other hand, the liability growth rate further moderates this behavior. Under a low liability growth rate, investors face lighter financial pressure and exhibit higher risk tolerance, leading to more aggressive strategies (such as high-frequency position adjustments) to capture market opportunities. In contrast, under a high liability growth rate, long-term solvency pressure compels investors to prioritize risk control to avoid exacerbating the gap between net assets and liabilities due to market volatility.

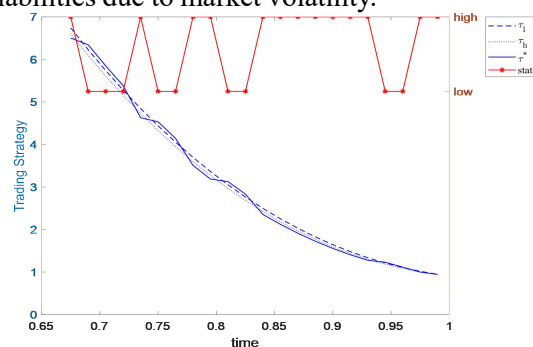


Figure 3. Optimal Trading Strategy τ^*
(Market Resilience Rate)

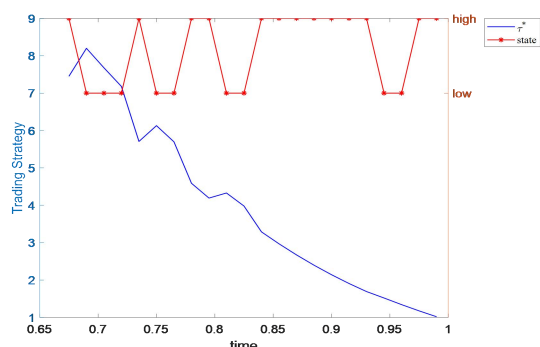


Figure 4. Optimal Trading Strategy under Constant Liquidity Regime τ^* (Only Liability Growth Rate Subject to State Transitions)

Analysis under both states indicates that the liability growth rate influences the trading strategy by altering risk tolerance and solvency pressure, while market liquidity, as a given environmental factor, determines the manifestation of this influence (such as the degree of aggressiveness or conservatism).

6. Conclusion

This paper constructs a dynamic Asset-Liability Management (ALM) model that integrates Markov regime switching, market frictions, and liability management, systematically investigating the optimal ALM strategies for financial institutions in complex market environments. Theoretical analysis demonstrates that the optimal trading strategy can be determined via the solution to a system of coupled Riccati differential equations. This strategy exhibits a state-dependent feedback form, enabling a dynamic trade-off between risk hedging and return objectives in stochastic market settings.

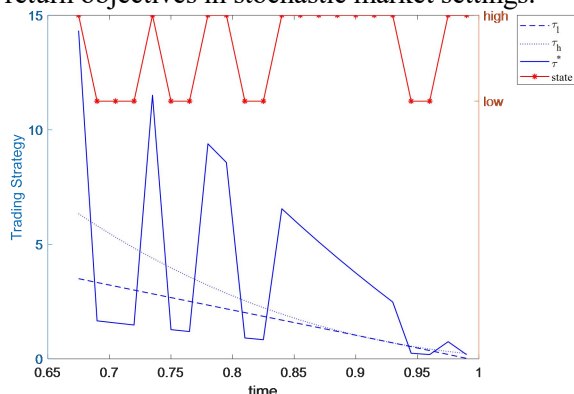


Figure 5. Optimal Trading Strategy τ^* under Regime-Switching Liquidity (Both Liability Growth Rate and Liquidity Parameters Subject to State Transitions)

Numerical simulations reveal that the liability growth rate independently influences investor

behavior, distinct from the market liquidity state. A high liability growth rate compels investors to adopt conservative strategies to control solvency risk, whereas a low liability growth rate incentivizes the pursuit of excess returns. Specifically, under a combination of low liquidity and high liability growth rate, the joint effect of market frictions and liability pressure further reinforces risk-averse behavior. Conversely, in states characterized by high liquidity and low liability growth rate, investors are more inclined to adjust positions actively to capture return opportunities.

This study elucidates trader behavior under varying liability rates and market liquidity conditions, providing theoretical support for financial institutions to optimize asset allocation and hedge liability risks in regime-switching market environments. Future work could extend this framework to more complex scenarios involving multiple asset classes, non-Gaussian noise, and other realistic market features.

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