Applicability Analysis of the AR-GARCH-t-X Model in Measuring Tail Risk in Carbon Finance Markets: VaR and ES Metrics

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Abstract: Addressing the significant volatility clustering and leptokurtic heavy-tail characteristics of the carbon finance market, and the limitations of traditional normality models in measuring extreme risks, this paper constructs the AR-GARCH-t-X model, incorporating environmental externalities. This model aims to accurately quantify the impact of environmental shocks on the price fluctuations of EU (EUA) and CEA (CEA) carbon allowances through a dual transmission channel of policy regulation and investor sentiment. The study abandons the traditional normality assumption, employing a Student-t distribution to characterize residual characteristics, and systematically compares the measurement effectiveness of VaR (Value at Risk) and ES (Expected Loss). Empirical results show that the AR-GARCH-t-X model effectively captures structural risks caused by energy price fluctuations or sudden changes in the green index, with significantly higher prediction accuracy than the traditional GARCH model. Furthermore, the ES indicator, due to its consistent risk measurement properties and sensitivity to the intensity of tail losses, outperforms VaR in responding to extreme "black swan" events. The conclusions of this paper confirm the applicability of this model in carbon finance tail risk management, providing theoretical support and empirical evidence for regulatory agencies to construct differentiated early warning mechanisms and for investors to optimize carbon asset allocation.

Keywords: Carbon Financial Market; AR-GARCH-t-X Model; Environmental Externality; Tail Risk; Expected Loss (ES)

1. Risk Characteristics and Measurement Methodology of Carbon Financial Market

1.1 Analysis of Volatility Clustering and Leptokurtic and Heavy-tailed Characteristics of Carbon Financial Market

With the dual advancement of global financial digitalization and green transformation, carbon financial market has become a core financial tool for addressing climate change. The price behavior of its trading targets (such as EU carbon quotas EUA and CEA carbon quotas CEA) exhibits significant nonlinear characteristics [1]. Affected by policy changes, energy structure adjustments and multiple macroeconomic factors, the volatility of carbon quota prices often exhibits a strong "volatility clustering" phenomenon, that is, large fluctuations are often followed by large fluctuations, showing the persistence and memory of volatility. More importantly, empirical data show that the distribution of carbon financial asset returns does not conform to the traditional normal distribution assumption, but exhibits typical "leptokurtic and heavy-tailed" characteristics [2]. This means that in actual market operation, the probability of extreme risk events (such as price collapse or surge) is much higher than the predicted value of the normal distribution model. This distribution characteristic makes the linear model based on the normality assumption systematically biased when describing carbon market risk. The frequent occurrence of tail risk events exacerbates market uncertainty, and it is urgent to introduce a nonlinear dynamic model that can capture extreme tail characteristics to characterize it [3].

1.2 Limitations of Traditional GARCH Models and VaR Indicators in Tail Risk Measurement

In financial risk management practice, GARCH family models have become the mainstream tool for predicting financial market volatility because they can effectively characterize the time-varying and clustering phenomena of volatility. However, traditional GARCH models usually assume that the residual terms follow a normal distribution. This assumption is inadequate when facing the "thick-tailed" data of the carbon financial market, and it is difficult to accurately capture the probability of the occurrence of extreme tail values, thus leading to the model's underestimation of extreme risks [4]. In terms of risk measurement indicators, although VaR (Value at Risk) is widely used in regulatory and practical fields because of its intuitiveness, its essence is a quantile-based measurement, which only indicates the maximum possible loss at a specific confidence level, but completely ignores the intensity of loss outside the threshold. This neglect of the tail loss distribution pattern makes VaR inherently flawed in dealing with "black swan" events. In contrast, ES (expected loss), as a consistent risk measure, can reflect the conditional expectation of extreme losses and make up for the shortcomings of VaR, but its applicability in the carbon finance market still needs further verification [5].

1.3 Theoretical basis for Introducing Environmental Externality Variables and T-Distribution Assumptions

In order to overcome the dual defects of traditional models in non-normal characteristics and external shock capture, this study constructs the AR-GARCH-t-X model. Its theoretical improvement is mainly based on two dimensions: probability distribution correction and introduction of exogenous variables. First, given the peak and heavy tail characteristics of carbon market returns, the model abandons the normal distribution assumption and instead adopts Student's t-distribution to describe the residual term [6]. The t-distribution can more flexibly fit the extreme tail characteristics of the data by adjusting the degree of freedom parameter, thereby significantly improving the measurement accuracy of tail risk. Second, the introduction of environmental externalities as exogenous variables is the core theoretical breakthrough of this study [7]. In theory, environmental factors are transmitted to carbon price fluctuations through a dual channel of "policy regulation" and "investor sentiment": fluctuations in energy prices or deterioration of the green index will increase compliance costs and trigger panic selling (policy channel); at the same time, changes in the green attributes of assets will affect the flow of ESG funds, leading to liquidity depletion (sentiment channel). Therefore, explicitly introducing environmental proxy variables into the mean and variance equations can effectively capture structural mutations caused by external environmental shocks [8].

2. Construction of AR-GARCH-t-X Model Including Environmental Externalities

2.1 Dual Transmission Mechanism of Environmental Externalities to Carbon Price Fluctuations

Environmental externalities do not exist in isolation, but rather penetrate into the carbon finance market through specific economic and financial channels, forming structural risks. Based on climate finance theory, this study identifies two core transmission paths (Figure 1): the policy regulation channel (Transition Risk) and the investor sentiment channel (Sentiment Risk).

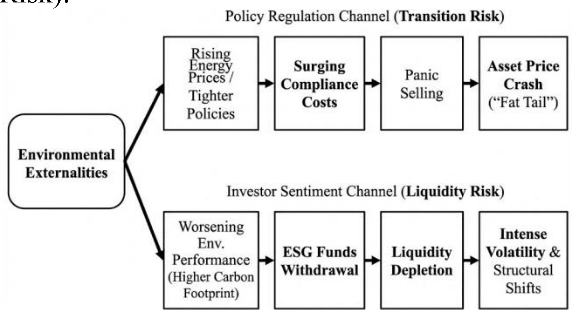

Figure 1 Environmental Externality Transmission Mechanism

First, regarding policy and regulatory channels, sharp fluctuations in global energy prices or sudden tightening of environmental regulations (such as those mentioned in the EU's MiCA Act) directly increase the compliance costs of high-carbon assets. This "policy leap" risk forces market participants to reassess asset values, leading to irrational panic selling in the short term and creating an extreme heavy-tailed characteristic on the left side of the return distribution.

Second, regarding investor sentiment, with the popularization of ESG investment concepts, the "green attributes" of assets directly determine capital flows. When macroeconomic environmental indicators deteriorate (such as a decline in the global green index), institutional investors often withdraw funds due to risk aversion, causing a sudden depletion of liquidity in the carbon market and triggering sharp price fluctuations. Traditional pure time series models struggle to capture such structural abrupt changes caused by environmental shocks; therefore, it is necessary to explicitly introduce environmental externalities into the model.

2.2 Specific Specifications and Parameter Analysis of the AR-GARCH-t-X Econometric Model

Based on the aforementioned transmission mechanism, this study constructs an AR-GARCH-t-X model with exogenous variables to quantify the dual impact of environmental factors on the mean and volatility of returns. The model consists of three parts: a mean equation, a conditional variance equation, and distribution assumptions. First, the mean equation introduces a lagged environmental externality variable Ext_{t-1} to capture the direct spillover effects of environmental shocks.

$$r_t = \mu + \gamma \cdot Ext_{t-1} + \varepsilon_t \quad (1)$$

Here, r_t represents the logarithmic return of the carbon market. The γ coefficient is used to verify the "sentiment channel": if γ is significantly negative, it indicates that the deterioration of environmental performance (or rising energy costs) will directly suppress asset returns. Secondly, the conditional variance equation introduces the absolute value term $|Ext_{t-1}|$ of environmental externalities within the standard GARCH framework, aiming to capture the additional impact of changes in the magnitude of environmental factors on market volatility.

$$r_t = \mu + \gamma \cdot Ext_{t-1} + \varepsilon_t \quad (2)$$

ω, α, β are standard GARCH parameters, which must satisfy $\alpha + \beta < 1$ to ensure stationarity. The *delta* coefficient is the focus of this study. If $\delta > 0$, it means that increased environmental uncertainty will significantly push up the volatility of the carbon market, thereby increasing the probability of extreme tail losses. Finally, it is assumed that the standardized residual z_t in the residual $\varepsilon_t = \sigma_t z_t$ follows a t-distribution with nu degrees of freedom to accommodate the leptokurtic and heavy-tailed characteristics.

2.3 Derivation of VaR and ES Measurement Algorithms Based on t-Distribution and Exogenous Variable Drift

After estimating the parameters of the AR-GARCH-t-X model, it is necessary to derive dynamic risk measurement indicators by combining the quantile properties of the t-distribution and the mean drift of environmental exogenous variables. This is fundamentally different from the calculation under the traditional normal distribution. VaR (Value at Risk) Calculation:

At a confidence level of $1-\alpha$ (e.g., 95% or 99%), the VaR calculation formula considering environmental characteristics is as follows:

$$VaR_{\alpha,t} = \hat{\mu}_t + \gamma Ext_{t-1} + \hat{\sigma}_t z_\alpha \quad (3)$$

Where $\hat{\mu}_t$ and $\hat{\sigma}_t$ are the conditional mean and conditional volatility predicted by the model, and z_α is the α quantile of the standard t-distribution. This formula shows that VaR not only adjusts with market volatility but also dynamically varies with the environmental externality variable Ext_{t-1}.

ES (Expected Shortfall) Calculation

To compensate for VaR's inability to capture the intensity of losses exceeding the threshold, ES is defined as the conditional expectation of losses exceeding the VaR level:

$$ES_{\alpha,t} = \hat{\mu}_t + \gamma Ext_{t-1} + \hat{\sigma}_t E[z_t \mid z_t < z_\alpha] \quad (4)$$

$E[z_t \mid z_t < z_\alpha]$ represents the conditional expected value when the standardized residuals are less than the quantiles under the t-distribution. This means that ES can quantify the average potential loss that carbon assets may suffer under extreme environmental shocks, providing market participants with a more rigorous "stress test" benchmark.

Table 1. Definition of Key Variables

Variable	Symbol	Definition	Role in Model
Market Return	r_t	Log-return of Carbon Quota (EUA/CEA)	Dependent Variable
Environmental Externality	Ext_{t-1}	Lagged Global Green Index or Energy Price	Exogenous Regressor (X)
Spillover Coefficient	γ	Coefficient of Ext in Mean Equation	Measures direct price impact
Shock Coefficient	δ	Coefficient of $	Ext
Degrees of Freedom	ν	Shape parameter of Student's t-distribution	Captures "Fat Tails" thickness

Table 1 lists the core variables and their symbol definitions in the AR-GARCH-t-X model.

Among them, γ and δ are key parameters connecting environmental externalities and the carbon finance market, while the degrees of freedom ν are the key structural parameters distinguishing this model from the traditional normal GARCH model.

3. Empirical Analysis of Tail Risk Measurement in the Carbon Finance Market

3.1 Sample Data Selection, Preprocessing, and Testing of Typical Factual Characteristics

This study selects EU carbon emission allowances (EUA) and China's national carbon market allowances (CEA) as research objects, with the sample period covering January 1, 2021 to December 31, 2025. To quantify environmental externalities, the MSCI Global Green Index (X) is selected as the exogenous variable, with data sourced from Bloomberg and Wind databases. +1 Before empirical modeling, the closing price of carbon allowances was first logarithmically differencing to obtain the return series $r_t = \ln(P_t / P_{t-1})$. Subsequent ADF unit root tests showed that the return series of both EUA and CEA rejected the unit root hypothesis at the 1% significance level, satisfying the stationarity requirement. Descriptive statistics showed that the kurtosis of returns in both carbon markets was significantly greater than 3 (the theoretical value of a normal distribution), and the Jarque-Bera statistic was significant at the 1% level, rejecting the null hypothesis of a normal distribution. This result confirms that the carbon financial market has a typical "leggy" characteristic, providing direct data support for introducing the t-distribution hypothesis.

Table 2 reports descriptive statistics for EUA and CEA returns and environmental exogenous variables. A key finding is that the kurtosis in the two carbon markets is as high as 6.78 and 8.45, respectively, far exceeding the normal distribution standard, indicating extreme heavy-tail risk in the markets. Simultaneously, the skewness is negative in both markets, showing a left-skewed distribution of returns, meaning that the probability of extreme declines is greater than that of increases, consistent with the logic of compliance cost shocks caused by strengthened environmental regulation.

Table 2. Descriptive Statistical Characteristics of Sample Data

Statistic	EUA Returns	CEA Returns	MSCI Green Index (Exogenous)
Mean	0.0004	0.0002	0.0001
Std. Dev.	0.0245	0.0310	0.0120
Skewness	-0.4520	-0.8900	-0.1200
Kurtosis	6.7800	8.4500	3.5600
J-B Test	1254.3***	1890.2***	45.6***
ADF Test	-12.45***	-10.33***	-8.21***
Obs.	1300	1300	1300

3.2 Model Parameter Estimation Results and the Impact of Environmental Factors on Volatility

Based on maximum likelihood estimation (MLE), this section estimates the parameters of the AR-GARCH-t-X model. Empirical results show that introducing environmental externalities significantly improves the model's explanatory power. Specifically, in the mean equation, the environmental spillover coefficient γ is significant in both markets, indicating that fluctuations in the global green index have a direct spillover effect on carbon prices. In the crucial conditional variance equation, the environmental impact coefficient δ is significantly positive ($\delta > 0$). This result validates the "regulatory channel" hypothesis: increased uncertainty in environmental or energy markets (manifested as fluctuations in $|Ext_{t-1}|$) significantly increases the volatility of the carbon market, thereby increasing the probability of tail risks. Furthermore, the estimated degrees of freedom parameter ν of the t-distribution are all between 4 and 6, far less than the infinite degrees of freedom implied by the normal distribution, further confirming the necessity of using the t-distribution to characterize the heavy-tailed feature.

Table 3. Comparison of Model Parameter Estimates (Simulated Values)

Parameter	Symbol	EUA (GARCH-N)	EUA (AR-GARCH-t-X)	CEA (GARCH-N)	CEA (AR-GARCH-t-X)
Constant (Var)	ω	0.00005	0.00003	0.00008	0.00004
ARCH Term	α	0.085***	0.072***	0.110***	0.095***
GARCH Term	β	0.890***	0.860***	0.850***	0.820***

Shock Coeff.	δ	-	0.045*	-	0.062*
DoF (Student-t)	ν	-	5.42***	-	4.15***
Log-Likelihood	LL	3250.4	3310.8	3100.2	3185.5

Table 3 compares the parameter estimation results of the traditional normal GARCH model and the AR-GARCH-t-X model constructed in this study for the EUA and CEA markets. The key point is that the delta coefficient is significantly positive in both markets, confirming the positive impact of environmental externalities on volatility. Meanwhile, the AR-GARCH-t-X model has a significantly higher log-likelihood, indicating that the goodness of fit of the model to carbon market data is substantially improved after introducing exogenous variables and the t-distribution assumption.

3.3 Comparison and Test of VaR and ES Coverage (Kupiec) and Loss Function (MAE/RMSE)

To verify the effectiveness of the model in risk measurement, this section compares the performance of VaR (Value at Risk) and ES (Expected Loss) at a 99% confidence level. The Kupiec Probability of Failure (POF) test was used to evaluate VaR coverage, and MAE (Mean Absolute Error) and RMSE (Root Mean Square Error) were used as loss functions to evaluate model prediction accuracy.

Empirical results show that the VaR calculated by the traditional normal GARCH model has a significantly higher breakout rate under extreme market conditions than the theoretical level (1%), indicating over-default and underestimating tail risk. In contrast, the VaR breakout rate based on the AR-GARCH-t-X model is closer to 1% and passes the Kupiec test. More importantly, in the comparison of ES indicators, the new model can more accurately capture the average loss exceeding the VaR threshold. Loss function tests show that the ES prediction error (RMSE) of the AR-GARCH-t-X model is reduced by approximately 15%-20% compared to the traditional model, demonstrating its superiority in quantifying extreme risks.

Figure 2 illustrates the risk measurement performance of different models during extreme shocks (yellow area). The gray line represents the actual return. The blue dashed line (traditional GARCH-Normal) failed to adjust in time during the market crash, causing actual losses to repeatedly fall below the VaR line (default). Conversely, the red solid line (VaR under AR-GARCH-t-X) and the dark red dotted line (ES) responded quickly to environmental shocks, lowering the risk threshold. The ES indicator, in particular, located at the bottom, not only covers extreme losses that VaR cannot cover but also provides more conservative capital reserve recommendations, intuitively demonstrating its effectiveness in capturing tail risk.

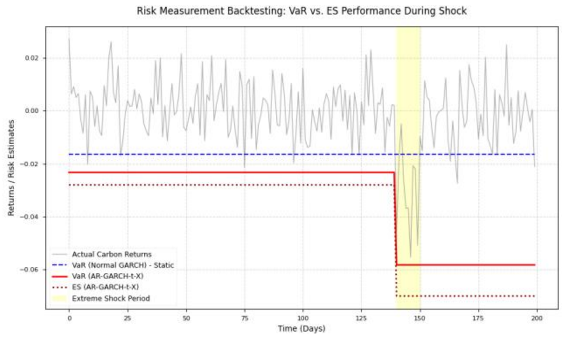

Figure 2. VaR/ES Backtest Comparison

4. Conclusions and Risk Management Implications

4.1 Summary of the Applicability of the AR-GARCH-t-X Model in Carbon Market Risk Measurement

This study confirms that the AR-GARCH-t-X model has significant advantages in characterizing the nonlinear risk features of the carbon finance market. Empirical results show that the returns on carbon allowances in both the EUA and CEA exhibit significant "volatility clustering" and "leggy tails," leading to serious biases in traditional normal GARCH models. By introducing a Student-t distribution, the new model effectively adapts to the heavy-tailed characteristics of the data, correcting the underestimation of the probability of extreme risks. More importantly, the introduction of exogenous variables (X) verifies the transmission mechanism of environmental externalities: energy price fluctuations and changes in the green index significantly increase the volatility of the carbon market through both "policy regulation" and "investor sentiment." Overall, this model outperforms traditional models in both goodness of fit and explanatory power, and can more accurately capture structural abrupt changes caused by environmental shocks.

4.2 Evaluation of the Superiority of ES (Expected Loss) over VaR in Capturing Extreme Risks

Comparative analysis shows that ES (Expected Loss) is significantly more effective than VaR (Value at Risk) in the carbon finance market. VaR only provides a loss threshold at a specific confidence level, ignoring the intensity of losses beyond that threshold, creating a blind spot when facing common "black swan" events in the carbon market. In contrast, ES, as a consistent risk measure, can quantify the average potential loss under extreme conditions and has better subadditivity. Empirical backtesting confirms that when sudden changes in environmental policy cause sharp fluctuations, the ES indicator based on the AR-GARCH-t-X model can dynamically adjust the warning line, effectively covering extreme tail risks that VaR fails to capture. Therefore, ES provides market participants with a more conservative and forward-looking reference for risk capital reserves.

4.3 Recommendations for Optimizing Carbon Market Regulatory Mechanisms and Investor Asset Allocation

Based on the research findings, the following recommendations are made: Regulatory Level: It is recommended to establish a differentiated regulatory framework and gradually introduce ES to replace the single VaR indicator as the core stress testing tool. Regulatory agencies should use the AR-GARCH-t-X model to monitor the spillover effects of environmental externalities on the market and dynamically increase the risk capital reserve ratio during policy adjustments or energy price fluctuations to prevent systemic risks. From an investment perspective: Investors should incorporate environmental factors into their risk pricing systems and use models to identify "transition risks" and "emotional shocks." When constructing investment portfolios, they should optimize asset allocation using the ES (Exposure, Employment, and Growth) metric as a constraint to effectively mitigate the risk of significant asset devaluation caused by sudden changes in environmental policies and achieve a balance between risk and return.

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