H_∞ Filtering for Neural Networks with Time-Varying Delays and Quantized Outputs

Jing Han, Zi Wang, Ruoxian Wang
School of Electronical Information Engineering, Wanjiang University of Technology, Ma'anshan, Anhui, China

Abstract: Based on a neural network model, the problem of H_∞ filtering for systems with time-varying delays and quantized outputs was studied. The main objective is to develop a quantized H_∞ filter that ensures, on the one hand, that the filtering error system (FES) exhibits an asymptotically stable state in the absence of external disturbances, and on the other hand, that the system has H_∞ performance with zero initial conditions. The first step is to combine appropriate Lyapunov Krasovskii functionals with some inequality techniques, and through differentiation reasoning and other methods, provide sufficient criteria for the error system to achieve H_∞ performance. The second step is to propose a design scheme for the required H_∞ filter using several decoupling methods and a series of proof processes. In the final step, the effectiveness of the designed H_∞ filter method was confirmed through numerical simulation examples and simulation images. In summary, this work provides a systematic and computationally manageable method for designing quantized H_∞ filters for time-varying delay neural networks.

Keywords: Neural Network Model; Time-Varying Delay; Quantitative Output; H_∞ Performance; Lyapunov Krasovskii Functionals

1. Introduction

It is widely recognized that neural networks contain a large number of interconnected neurons that can transmit and store information. Due to its unique connection mechanism, the neural network exhibits features such as strong robustness, anti-distortion, adaptability, and self-learning capabilities. Hopfield [1] designed a content-addressable memory dynamic neural network model based on neurobiological knowledge. However, in a relatively large neural network, to effectively utilize the network, the state of neurons can be estimated from available measurements. Subsequently, design decisions can be made based on these estimates, leveraging them to achieve desired system performance, such as H_∞ performance or passivity performance. In reference [2], Yan et al. devoted themselves to developing a Luenberger observer that enables the FES to achieve a predetermined exponential level of H_∞ interference suppression in the event of possible sensor failures. Tai [3] et al. considered a class of switching neural network systems with time delay and reaction-diffusion terms, and developed a design method using Lyapunov Krasovskii functionals and some inequalities. This method not only makes the FES exponentially stable, but also forces it to be in a passive state under the influence of external disturbances on the output error. Lee [4] et al. proposed a new output feedback fuzzy control design technique based on sampled data L_2-L_∞ filters for active suspension systems subject to hard constraints. Using input delay method to solve the sampling data problem of continuous time suspension system. Zhou [5] et al. discussed the quantization asynchronous filtering problem of a semi-Markov switching system described by a Takagi Sugeno fuzzy model with multiple random delays under deception attacks. Cai [6] et al. designed a filter to ensure that the FES satisfies both H_∞ performance and passive performance indicators in a finite time domain. Liu [7] et al. studied the H_∞ filtering of continuous time nonlinear systems described by Takagi Sugeno (T-S) fuzzy models with quantization effects, and developed a novel filter that combines dynamic quantization mechanisms to ensure that the quantized closed-loop system maintains the specified performance. But in practical engineering, the output of neural networks must pass through an uncertain communication link to reach the filter end. Due to limited network bandwidth, it may cause phenomena such as information loss and information disorder. Therefore, by using quantization techniques to reduce communication rates and performing quantization processing on output signals, such problems can be solved to a certain extent. This inspires the study of the H_∞ filtering problem for time-delay neural networks with quantized outputs in this paper.

This article systematically studies the H_∞ filtering problem of a class of neural networks that simultaneously have time-varying delays and quantized outputs. The research objective is to design an H_∞ filter that includes a quantization step to ensure that the FES can meet the given H_∞ performance indicators with zero initial conditions. Firstly, by selecting appropriate Lyapunov Krasovskii functionals and utilizing several recently developed integral inequality techniques, a sufficient criterion has been established for the H_∞ performance of the filtering error system. Secondly, based on this criterion, supplemented by various decoupling methods, the specific design steps of the H_∞ filter are provided. Ultimately, an example was used to verify the applicability of the designed quantized H_∞ filter in suppressing interference and quantization errors.

2. Prerequisite Knowledge

The symbols involved in the subsequent content are defined in Table 1.

Table 1. Symbol Table

Symbol	Explanation
$\mathbb{R}^{n\times m}$	real matrix with n rows and columns
$\mathbb{R}^n$	n-dimensional Euclidean space
S_+^n	n- dimensional positive definite matrix
S^n	$n\times n$ real symmetric matrix
B^{-1} and B^T	inverse matrix of B and transpose matrix of B
$He(B)$	$B+B^T$
$\|x\|$	Euclidean norm of vector x
I	appropriate dimensional identity matrix
$L_2[0,\infty)$	the space of square integrable infinite-dimensional vector functions over interval $[0,\infty)$
ς_i	n-dimensional column vector, the i-th row is 1 and the remaining rows are all 0

2.1 System Model

In reference [8], a network system model under time delay invariance was considered, and a quantized H_∞ filter was studied and designed to assure that the FES meets H_∞ performance with zero initial conditions. But in practical neural networks, time delay is not fixed, but may vary over time. Therefore, when the time delay in the system varies, the following neural network system model is adopted:

$$\dot{s}(\tau)=Es(\tau)+F_1\pi\big(s(\tau)\big)+F_2\pi\big(s(\tau-h(\tau))\big)+J_1\varpi(\tau), \quad (1)$$

$$o(\tau)=G_1s(\tau)+G_2s\big(\tau-h(\tau)\big)+J_2w(\tau), \quad (2)$$

Where $s(\tau)=[s_1(\tau)\cdots s_n(\tau)]^T\in\mathbb{R}^n$ means the neuron state, $o(\tau)=[o_1(\tau)\cdots o_m(\tau)]^T\in\mathbb{R}^m$ means the system output and $\varpi(\tau)\in\mathbb{R}^m$ is an L_2 type external interference input defined on $[0,\infty)$; $E=\mathrm{diag}\{-e_1,\cdots,-e_n\}\in\mathbb{R}^{n\times n}$ is a self feedback matrix with $e_i>0,(i=1,\dots,n)$; $F_1\in\mathbb{R}^{n\times n}$和$F_2\in\mathbb{R}^{n\times n}$ is a connection weight matrix; $J_1\in\mathbb{R}^{n\times m}, G_1\in\mathbb{R}^{m\times n}, G_2\in\mathbb{R}^{m\times n}, J_2\in\mathbb{R}^{m\times m}$ are known parameter matrices; $\pi\big(s(\tau)\big)=\big[\pi_1\big(s(\tau)\big)\cdots\pi_n\big(s(\tau)\big)\big]^T\in\mathbb{R}^n\to\mathbb{R}^n$, represents the activation function that satisfies the Lipschitz condition [9] and there have a Lipschitz constant $L_\pi>0$, which satisfies inequality:

$$\|\pi(c_2)-\pi(c_1)\|\le L_\pi\|c_2-c_1\|, \forall c_1,c_2\in\mathbb{R}^n. \quad (3)$$

The variable delay is represented by $h(\tau)$, and the following inequality holds:

$$0<h_1<h(\tau)<h_2,\ h_{12}\triangleq h_2-h_1. \quad (4)$$

After considering the role of output quantization, the proposed filter structure can be described by the following equation:

$$\dot{\check{s}}(\tau)=E\check{s}(\tau)+F_1\pi\big(\check{s}(\tau)\big)+F_2\pi\big(\check{s}(\tau-h(\tau))\big)+LZ[o(\tau)-\check{o}(\tau)], \quad (5)$$

$$\check{o}(\tau)=G_1\check{s}(\tau)+G_2\check{s}(\tau-h(\tau)), \quad (6)$$

Where $\check{s}(\tau)=[\check{s}_1(\tau)\cdots\check{s}_n(\tau)]^T\in\mathbb{R}^n$ indicates the state of the filter, $\check{o}(\tau)=[\check{o}_1(\tau)\cdots\check{o}_m(\tau)]^T\in\mathbb{R}^m$ represents the output of the filter; $L\in\mathbb{R}^{n\times m}$ is the filter gain to be calculated; In addition, the logarithmic quantizer is represented by $Z(y)=[Z_1(r_1)\cdots Z_m(r_m)]^T$ and satisfies

$$Z_f(y_f)=\begin{cases}\chi^l\psi_0, & \frac{1}{1+\rho}\chi^l\psi_0<y_f\le\frac{1}{1-\rho}\chi^l\psi_0, l=0,\pm1,\pm2,\dots\\ 0, & y_f=0,\\ -Z_f(-y_f), & y_f<0,\end{cases} \quad (7)$$

for any $f\in\{1,\dots,m\}$, where $\psi_0>0$ is a scaling scalar, where $\rho=\frac{1-\chi}{1+\chi}(0<\chi<1)$ denotes the quantization density [10].

Using the sector boundary method, $Z(y)$ can be written as

$$Z(y)=(1+\delta)y, \quad (8)$$

Where $\delta\in[-\rho,\rho]$, From equation (8), the following FES can be established:

$$\dot{\kappa}(\tau) = \overline{E}_1\kappa(\tau) - \overline{E}_2\kappa\big(\tau - h(\tau)\big) + F_1 d\big(\kappa(\tau)\big) + F_2 d\big(\kappa(\tau - h(\tau))\big) + \overline{J}\varpi(\tau), \quad (9)$$

$$\tilde{o}(\tau) = G_1\kappa(\tau) + G_2\kappa(\tau - h(\tau)) + J_2\varpi(\tau), \quad (10)$$

with

$$\kappa(\tau) = s(\tau) - \check{s}(\tau), \quad \tilde{o}(\tau) = o(\tau) - \check{o}(\tau),$$
$$d\big(\kappa(\tau)\big) = \pi\big(\kappa(\tau) + \check{s}(\tau)\big) - \pi\big(\check{s}(\tau)\big),$$
$$d\big(\kappa(\tau - h(\tau))\big) = \pi\big(\kappa(\tau - h(\tau)) + \check{s}(\tau - h(\tau))\big) - \pi\big(\check{s}(\tau - h(\tau))\big),$$
$$\overline{E}_1 = E - (1+\delta)LG_1, \overline{E}_2 = (1+\delta)LG_2, \overline{J} = J_1 - (1+\delta)LJ_2.$$

We construct a quantization filter (as shown in equations (5) - (6)) for the neural network described by equations (1) - (2), with the goal of making the FES (9) - (10) have H_∞ performance. In other words:

Definition 1: Assuming that $\gamma > 0$, is given and the initial condition is set to zero, when the following equation holds:

$$\int_0^\infty \kappa^T(x)\,\Upsilon\kappa(x)dx \le \gamma^2 \int_0^\infty \varpi^T(x)\varpi(x)\,dx, \forall\, x \ge 0. \quad (11)$$

So, it is called that the FES (9) - (10) has reached the H_∞ interference attenuation level γ, where Υ represents a known positive definite symmetric matrix.

Remark 1: It is worth noting that we can define

$$H(\tau) = \sqrt{\frac{\int_0^\tau \kappa^T(x)\,\Upsilon\kappa(x)dx}{\int_0^\tau \varpi^T(x)\varpi(x)\,dx}}, \quad (12)$$

Thus, equation (10) can be rewritten as $H(\tau) \le \gamma$, where γ is the interference attenuation index and also represents H_∞ norm boundedness.

Remark 2: For the first time, the H_∞ performance metric was introduced into the design of filters and controllers in reference [11]. The core idea of this method is to first establish a transfer function between random perturbations and estimation errors, and then minimize the maximum value of this transfer function (i.e. its infinite norm) so that it does not exceed a predetermined upper bound.

Lemma 1 [12]: Given matrix $X \in S_+^n$, for the differentiable function $\eta \in \mathbb{R}^n$ on the interval $[n, m]$, the following inequality holds:

$$\int_n^b \dot{\eta}^T(u)X\dot{\eta}(u)du \ge \frac{1}{b-n}\Omega^T diag\{X, 3X, 5X\}\Omega \quad (13)$$

with

$$\Omega = \begin{bmatrix} \eta(m) - \eta(n) \\ \eta(m) + \eta(n) - \frac{2}{m-n}\int_n^m \eta(x)dx \\ \eta(m) - \eta(n) - \frac{6}{m-n}\int_n^b \xi_{n,m}(x)\eta(x)dx \end{bmatrix}, \xi_{a,b}(x) = 2\left(\frac{x-n}{m-n}\right) - 1.$$

Lemma 2 [13]: If Φ_1 and Φ_2 are real matrices with arbitrary dimension matching, then there exists a constant r greater than zero, which satisfies the following condition:

$$\Phi_1\Phi_2^T + \Phi_2\Phi_1^T \le \frac{1}{r}\Phi_1\Phi_1^T + r\Phi_2\Phi_2^T. \quad (14)$$

Lemma 3 [14]: (Convex Combination Inequality) For any given matrix $R \in S_+^n$, assuming a matrix $C \in \mathbb{R}^{n\times n}$ satisfies the condition:

$$\begin{bmatrix} R & C \\ * & R \end{bmatrix} \ge 0, \quad (15)$$

then there is

$$\begin{bmatrix} \frac{1}{\alpha}R & 0 \\ 0 & \frac{1}{1-\alpha}R \end{bmatrix} \ge \begin{bmatrix} R & X \\ * & R \end{bmatrix}, \forall \alpha \in (0,1).$$

3. Main Results

Theorem 1 Given a parameter $\gamma > 0$ and a known matrix Υ, assuming the existence of matrices $P = (P_{ij})_{5\times5} \in S_+^{5n}, S_1, S_2, R_1, R_2 \in S_+^n, U_1, U_2 \in \mathbb{R}^{n\times n}, N_1, N_2 \in \mathbb{R}^{18n\times 2n}$, diagonal matrices $Q_1, Q_2 \in S_+^n$, and matrix $X \in \mathbb{R}^{3n\times 3n}$, the following inequality holds

$$\Psi = \begin{bmatrix} \widetilde{R}_2 & X \\ X^T & \widetilde{R}_2 \end{bmatrix} \ge 0, \quad (16)$$

$$\Pi(h(\tau)) = \Pi_0(h(\tau)) - \Gamma^T\Psi\Gamma < 0, \quad (17)$$

for any $h(\tau) \in [h_1, h_2]$, where

$$\begin{aligned}\Pi_0(h(\tau)) = {} & He\big[T_1^T(h(\tau))PT_0 + N_1\beta_1\big(h(\tau)\big) \\ & + N_2\beta_2\big(h(\tau)\big) + T_6^T\widehat{U}T_7\big] + \check{S}_1 \\ & + \varsigma_{18}^T\big(h_1^2R_1 + h_{12}^2R_2\big)\varsigma_{18} - T_2^T\widetilde{R}_1T_2 \\ & + T_5^T\widehat{S}_2T_5 + \varsigma_1^T\Upsilon\varsigma_1 - \gamma^2\varsigma_{17}^T\varsigma_{17},\end{aligned}$$

$$\widehat{S}_2 = diag\{L_\pi^2Q_1, L_\pi^2Q_2, -Q_1, -Q_2\},$$

$$\check{S}_1 = diag\{2S_1, -S_1 + S_2, 0_{n\times n}, -S_2, 0_{14n\times 14n}\},$$

$$\widehat{U} = [U_1 \quad U_2]^T,$$

$$\widetilde{R}_i = diag\{R_i, 3R_i, 5R_i\}, i = 1,2,$$

$$T_0 = \begin{bmatrix}\varsigma_{18}^T & \varsigma_1^T - \varsigma_2^T & \varsigma_1^T + \varsigma_2^T - 2\varsigma_5^T & \varsigma_2^T - \varsigma_4^T & \widehat{T}_0^T\end{bmatrix}^T,$$

$$\widehat{T}_0 = h_{12}(\varsigma_2 + \varsigma_4) - 2(\varsigma_{11} + \varsigma_{13}),$$

$$T_1\big(h(\tau)\big) = \begin{bmatrix}\varsigma_1^T & h_1\varsigma_5^T & h_1\varsigma_6^T & \varsigma_{11}^T + \varsigma_{13}^T & \widehat{T}_1^T(h(\tau))\end{bmatrix}^T,$$

$$\widehat{T}_1\big(h(\tau)\big) = \big(h_2 - h(\tau)\big)(\varsigma_{11} + \varsigma_{14}) + \big(h(\tau) - h_1\big)(\varsigma_{12} - \varsigma_{13}),$$

$$T_2 = [\varsigma_1^T - \varsigma_2^T \quad \varsigma_1^T + \varsigma_2^T - 2\varsigma_5^T \quad \varsigma_1^T - \varsigma_2^T - 6\varsigma_6^T]^T,$$

$$T_3 = [\varsigma_2^T - \varsigma_3^T \quad \varsigma_2^T + \varsigma_3^T - 2\varsigma_7^T \quad \varsigma_2^T - \varsigma_3^T - 6\varsigma_8^T]^T,$$

$$T_4 = [\varsigma_3^T - \varsigma_4^T \quad \varsigma_3^T + \varsigma_4^T - 2\varsigma_9^T \quad \varsigma_3^T - \varsigma_4^T - 6\varsigma_{10}^T]^T,$$

$$T_5 = [\varsigma_1^T \quad \varsigma_3^T \quad \varsigma_{15}^T \quad \varsigma_{16}^T]^T, \qquad G_6 =$$

$[\varsigma_{18}^T \quad \varsigma_1^T]^T$,

$$T_7 = \varsigma_{18} - \bar{E}_1\varsigma_1 + \bar{E}_2\varsigma_3 - F_1\varsigma_{15} - F_2\varsigma_{16} + \bar{J}\varsigma_{17},$$

$\Gamma = [G_3^T \quad G_4^T]^T, \beta_1(h(\tau)) = [h(\tau) - h_1]\begin{bmatrix}\varsigma_7\\ \varsigma_8\end{bmatrix} - \begin{bmatrix}\varsigma_{11}\\ \varsigma_{12}\end{bmatrix}$,

$\beta_2(h(\tau)) = [h_2 - h(\tau)]\begin{bmatrix}\varsigma_9\\ \varsigma_{10}\end{bmatrix} - \begin{bmatrix}\varsigma_{13}\\ \varsigma_{14}\end{bmatrix}$,

$\varsigma_i = [0_{n\times(i-1)n} \quad I_n \quad 0_{n\times(18-i)n}]^T, i = 1, ..., 18,$

This indicates that the error systems (9) - (10) can be regarded as having a γ level H_∞ anti-interference ability in the zero initial state.

Proof: Setting

$$\lambda(\tau) = col\{\phi_0(\tau), \phi_1(\tau), \phi_2(\tau), \phi_3(\tau), \phi_4(\tau), \phi_5(\tau), \phi_6(\tau)\}, \quad (18)$$

with

$$\phi_0(\tau) = \left[\kappa^T(\tau) \quad \kappa^T(\tau - h_1) \quad \kappa^T(\tau - h(\tau)) \quad \kappa^T(\tau - h_2)\right]^T,$$

$$\phi_1(\tau) = \frac{1}{h_1}\left[\int_{-h_1}^{0} \kappa_\tau^T(x)dx \quad \int_{-h_1}^{0} \xi_1(x)\kappa_\tau^T(x)dx\right]^T,$$

$$\phi_2(\tau) = \frac{1}{h(\tau)-h_1}\left[\int_{-h(\tau)}^{-h_1} \kappa_\tau^T(x)dx \quad \int_{-h(\tau)}^{-h_1} \xi_2(x)\kappa_\tau^T(x)dx\right]^T,$$

$$\phi_3(\tau) = \frac{1}{h_2-h(\tau)}\left[\int_{-h_2}^{-h(\tau)} \kappa_\tau^T(x)dx \quad \int_{-h_2}^{-h(\tau)} \xi_3(x)\kappa_\tau^T(x)dx\right]^T,$$

$\phi_4(\tau) = (h(\tau) - h_1)\phi_2(\tau), \phi_5(\tau) = (h_2 - h(\tau))\phi_3(\tau),$

$\phi_6(\tau) = \left[d^T(\kappa(\tau)) \quad d^T\left(\kappa(\tau - h(\tau))\right) \quad \varpi^T(\tau) \quad \dot{\kappa}^T(\tau)\right]^T$,

$$\kappa_\tau(x) = \kappa(\tau + x),$$
$$\xi_1(x) = 2\frac{x+h_1}{h_1} - 1,\ \xi_2(x) = 2\frac{x+h(\tau)}{h(\tau)-h_1} - 1,$$
$$\xi_3(x) = 2\frac{x+h_2}{h_2-h(\tau)} - 1,\ \xi_4(x) = 2\frac{x+h_2}{h_{12}} - 1.$$

It can be noted that $\beta_1(h(\tau))\lambda(\tau) = \beta_2(h(\tau))\lambda(\tau) = 0$, for any matrix $N_1, N_2 \in \mathbb{R}^{18n\times 2n}$, we can obtain the following equation:

$$2\lambda^T(\tau)[N_1\beta_1(h(\tau)) + N_2\beta_2(h(\tau))]\lambda(\tau) = 0. \quad (19)$$

Furthermore, for any matrix $U_1, U_2 \in \mathbb{R}^{n\times n}$ and any diagonal matrix $Q_1, Q_2 \in S^{n\times n}$, we can obtain

$$\lambda^T(\tau)He(T_6^T\widehat{U}T_7)\lambda(\tau) = 0, \quad (20)$$

$$\lambda^T(\tau)(T_5^T\hat{S}_2T_5)\lambda(\tau) \geq 0. \quad (21)$$

Introducing a new form of Lyapunov Krasovsky functional:

$$V(\kappa_\tau, \dot{\kappa}_\tau) = V_1(\kappa_\tau) + V_2(\kappa_\tau) + V_3(\kappa_\tau, \dot{\kappa}_\tau), \quad (22)$$

$$V_1(\kappa_\tau) = \int_{\tau-h_1}^{\tau} \kappa^T(x)S_1\kappa(x)dx + \int_{\tau-h_2}^{\tau-h_1} \kappa^T(x)S_2\kappa(x)dx,$$

$$V_2(\kappa_\tau) = \tilde{\kappa}^T(\tau)P\tilde{\kappa}(\tau),$$

$$V_3(\kappa_\tau, \dot{\kappa}_\tau) = h_1\int_{-h_1}^{0}\int_{\tau+\theta}^{\tau} \dot{\kappa}^T(x)R_1\dot{\kappa}(x)dxd\theta + h_{12}\int_{-h_2}^{-h_1}\int_{\tau+\theta}^{\tau} \dot{\kappa}^T(x)R_2\dot{\kappa}(x)dxd\theta,$$

with

$$\tilde{\kappa}(\tau) = col\{\kappa(\tau), h_1\phi_1(\tau), \phi_7(\tau)\}$$

$$\phi_7(\tau) = \left[\int_{-h_2}^{-h_1} \kappa_\tau^T(x)dx \quad \int_{-h_2}^{-h_1} \xi_4(x)\kappa_\tau^T(x)dx\right]^T.$$

By taking the derivative sequentially, the derivative of $V_1(\kappa_\tau)$ can obtain:

$$\dot{V}_1(\kappa_\tau) = \lambda^T(\tau)\hat{S}_1\lambda(\tau) - \kappa^T(\tau)S_1\kappa(\tau). \quad (23)$$

Due to

$$h_{12}\xi_4(x) = (h(\tau) - h_1)\xi_2(x) + (h_2 - h(\tau)),$$
$$h_{12}\xi_4(x) = (h_2 - h(\tau))\xi_3(x) + (h(\tau) - h_1), \quad (24)$$

So, we can write

$$\phi_7(\tau) = \begin{bmatrix}\varsigma_{11} + \varsigma_{13}\\ \widehat{T}_1(h(\tau))\end{bmatrix}\lambda(\tau). \quad (25)$$

By combining $\kappa(\tau) = \varsigma_1\lambda(\tau), h_1\phi_1(\tau) = h_1[\varsigma_5^T \quad \varsigma_6^T]^T\lambda(\tau)$, with the above expression, and then taking the derivative of $V_2(\kappa_\tau)$, we can calculate

$$\dot{V}_2(\kappa_\tau) = 2\tilde{\kappa}^T(\tau)P\dot{\tilde{\kappa}}(\tau) = \lambda^T(\tau)He[T_1^T(h(\tau))PT_0]\lambda(\tau). \quad (26)$$

In addition, by making use of Lemma 1 and Lemma 3, the derivative of $V_3(\kappa_\tau, \dot{\kappa}_\tau)$ can be derived as:

$$\dot{V}_3(\kappa_\tau, \dot{\kappa}_\tau) = h_1^2\dot{\kappa}^T(\tau)R_1\dot{\kappa}(\tau) + h_{12}^2\dot{\kappa}^T(\tau)R_2\dot{\kappa}(\tau) - h_1\int_{\tau-h_1}^{\tau} \dot{\kappa}^T(x)R_1\dot{\kappa}(x)dx - h_{12}\int_{\tau-h_2}^{\tau-h_1} \dot{\kappa}^T(x)R_2\dot{\kappa}(x)dx$$
$$\leq \lambda^T(\tau)[\varsigma_{18}^T(h_1^2R_1 + h_{12}^2R_2)\varsigma_{18} - T_2^T\widetilde{R}_1T_2 - \Gamma^T\Psi\Gamma]\lambda(\tau). \quad (27)$$

By combining equations (19)- (21), (23) and (26) - (27), we can easily derive the following inequality

$$\dot{V}(\kappa_\tau, \dot{\kappa}_\tau) \leq \lambda^T(\tau)[He(T_1^T(h(\tau))PT_0 + N_1\beta_1(h(\tau)) + N_2\beta_2(h(\tau)) + T_6^T\widehat{U}T_7) + \check{S}_1 + \varsigma_{18}^T(h_1^2R_1 + h_{12}^2R_2)\varsigma_{18} - T_2^T\widetilde{R}_1T_2 + T_5^T\hat{S}_2T_5 - \Gamma^T\Psi\Gamma]\lambda(\tau). \quad (28)$$

Next step, we will demonstrate the performance of H_∞. Define

$$K \triangleq \int_0^\tau \left(\kappa^T(x)\Upsilon\kappa(x) - \gamma^2\varpi^T(x)\varpi(x)\right)dx. \quad (29)$$

With the zero initial conditions, integrate $\dot{V}(\kappa_\tau, \dot{\kappa}_\tau)$ from 0 to τ, and according to the Newton Leibniz formula, it can be obtained:

$$\int_0^\tau \dot{V}(\kappa_x, \dot{\kappa}_x)dx = V(\tau) - V(0). \quad (30)$$

And because $V(\tau) > 0$, it can be concluded that

$$K = K + \int_0^\tau \dot{V}(\kappa_x, \dot{\kappa}_x)dx - (V(\tau) - V(0))$$
$$\leq K + \int_0^\tau \dot{V}(\kappa_x, \dot{\kappa}_x)dx = \int_0^\tau \lambda^T(x)\Pi(h(x))\lambda(x)dx. \quad (31)$$

By using inequality (17), we can conclude that $K \leq 0$, which means that

$$\int_0^\tau \kappa^T(x)\Upsilon\kappa(x)dx \leq \gamma^2\int_0^\tau \varpi^T(x)\varpi(x)\,dx, \quad (32)$$

Further, the following inequality can be derived:

$$\int_0^\infty \kappa^T(x)\Upsilon\kappa(x)dx \leq \gamma^2\int_0^\infty \varpi^T(x)\varpi(x)\,dx. \quad (33)$$

It is not difficult to see from this that equation (11) is satisfied, thus confirming that when the initial condition is zero, systems (9) - (10) have an H_∞ disturbance attenuation level γ with zero initial conditions.

Remark 3: The criterion for the H_∞ interference attenuation performance of quantized filtering error systems (9) - (10) is given by Theorem 1. It is not difficult to observe that when $\varpi(\tau) \equiv 0$, we can know

$$\dot{V}(\kappa_\tau, \dot{\kappa_\tau}) \leq -\kappa^T(\tau)\Upsilon\kappa(\tau), \quad (34)$$

Therefore, according to the Lyapunov stability theory, the criterion of Theorem 1 ensures that the FES (9) - (10) is asymptotically stable in the absence of external disturbances.

Based on Theorem 1, a design strategy based on linear matrix inequality (LMI) can be developed under varying time delays, as shown below

Theorem 2 Given parameters $\gamma > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $\rho > 0$ and a known matrix Υ, if there exists a scalar $\varepsilon > 0$, the matrices $P = (P_{ij})_{5\times5} \in S_+^{3n}$, S_1, S_2, R_1, $R_2 \in S_+^n$, N_1, $N_2 \in \mathbb{R}^{18n\times2n}$, $M \in \mathbb{R}^{n\times m}$, diagonal matrices Q_1, $Q_2 \in S_+^n$, and matrix $X \in \mathbb{R}^{3n\times3n}$ such that the following LMIs holds:

$$\Psi = \begin{bmatrix} \widetilde{R}_2 & X \\ X^T & \widetilde{R}_2 \end{bmatrix} \geq 0, \quad (35)$$

$$\begin{bmatrix} \Pi_1(h(\tau)) - \Gamma^T\Psi\Gamma & \rho W_1 \\ * & -\varepsilon I \end{bmatrix} < 0, \quad (36)$$

Where

$$\begin{aligned}\Pi_1(h(\tau)) = He\Big(&T_1^T(h(\tau))PT_0 + N_1\beta_1(h(\tau)) \\ &+ N_2\beta_2(h(\tau))\Big) + \widetilde{\Sigma}_1 + \widetilde{\Sigma}_2 + \check{S}_1 \\ &+ \varsigma_{18}^T(h_1^2R_1 + h_{12}^2R_2)\varsigma_{18} \\ &- T_2^T\widetilde{R}_1T_2 + T_5^T\hat{S}_2T_5 + \varsigma_1^T\Upsilon\varsigma_1 \\ &- \gamma^2\varsigma_{17}^T\varsigma_{17} + \varepsilon W_2W_2^T,\end{aligned}$$

$$\begin{aligned}\widetilde{\Sigma}_1 = He\big[&\varsigma_{18}^T(-\mu_1MG_1 + \mu_1P_{11}E)\varsigma_1 \\ &- \mu_1\varsigma_{18}^TMG_2\varsigma_3 + \mu_1\varsigma_{18}^TP_{11}F_1\varsigma_{15} \\ &+ \mu_1\varsigma_{18}^TP_{11}F_2\varsigma_{16} + \varsigma_{18}^T(-\mu_1MJ_2 \\ &+ \mu_1P_{11}J_1)\varsigma_{17} - \mu_1\varsigma_{18}^TP_{11}\varsigma_{18}\big],\end{aligned}$$

$$\begin{aligned}\widetilde{\Sigma}_2 = He\big[&\varsigma_1^T(-\mu_2MG_1 + \mu_2P_{11}E)\varsigma_1 \\ &- \mu_2\varsigma_1^TMG_2\varsigma_3 + \mu_2\varsigma_1^TP_{11}F_1\varsigma_{15} \\ &+ \mu_2\varsigma_1^TP_{11}F_2\varsigma_{16} + \varsigma_1^T(-\mu_2MJ_2 \\ &+ \mu_2P_{11}J_1)\varsigma_{17} - \mu_2\varsigma_1^TP_{11}\varsigma_{18}\big],\end{aligned}$$

$$W_1 = [-\mu_2M^T \quad 0_{16n\times16n} \quad -\mu_1M^T]^T,$$

$$W_2 = [G_1 \quad 0 \quad G_2 \quad 0_{13n\times13n} \quad J_2 \quad 0]^T,$$

Referring to Theorem 1 for other symbols, the system (5) - (6) when the gain matrix $L = P_{11}^{-1}M$ is the quantized H_∞ filter required for neural networks (1) - (2).

Proof: Let $U_1 = -\mu_1P_{11}, U_2 = -\mu_2P_{11}$. So, it makes the following equation holds:

$$\begin{aligned}T_6^T\widehat{U}T_7 = \widetilde{\Sigma}_1 + \widetilde{\Sigma}_2 &- \delta\mu_1\varsigma_{18}^TMG_1\varsigma_1 - \\ \delta\mu_1\varsigma_{18}^TMG_2\varsigma_3 - \delta\mu_1\varsigma_8^TMJ_2\varsigma_{17} &- \delta\mu_2\varsigma_1^TMG_1\varsigma_1 - \\ \delta\mu_2\varsigma_1^TMG_2\varsigma_3 &- \delta\mu_2\varsigma_1^TMJ_2\varsigma_{17}. \quad (37)\end{aligned}$$

According to the above equation, equation (37) can be rewritten as:

$$\widehat{\Pi}_1(h(\tau)) + He(\delta W_1W_2^T) < 0, \quad (38)$$

with

$$\begin{aligned}\widehat{\Pi}_1(h(\tau)) = He\big[&T_1^T(h(\tau))PT_0 + N_1\beta_1(h(\tau)) \\ &+ N_2\beta_2(h(\tau))\big] + \widetilde{\Sigma}_1 + \widetilde{\Sigma}_2 + \hat{S}_1 \\ &+ \varsigma_{18}^T(h_1^2R_1 + h_{12}^2R_2)\varsigma_{18} - T_2^T\widetilde{R}_1T_2 \\ &+ T_5^T\hat{S}_2T_5 + \varsigma_1^T\Upsilon\varsigma_1 - \gamma^2\varsigma_{17}^T\varsigma_{17}.\end{aligned}$$

Furthermore, From Lemma 2, it can be concluded that if

$$\widehat{\Pi}_1(h(\tau)) + \frac{\rho^2}{\varepsilon}W_1W_1^T + \varepsilon W_2W_2^T < 0. \quad (39)$$

So it can be proven that equations (9) - (10) hold true. Then, by setting $L = P_{11}^{-1}M$ and using Schur complement, we can reorganize equation (39) into equation (36). In this way, it is proved that the conditions in the theorem imply the conditions in Theorem 1, thereby ensuring the H_∞ performance of systems (9) - (10) under zero initial conditions.

4. Numerical Simulation Example

For the sake of verifying the applicability of the quantization filter design method in this article, a numerical example is provided and corresponding simulation analysis is conducted in this section.

The modeling of a dual neuron neural network is affected by time-varying delays and the following parameters:

$$E = \begin{bmatrix} -1 & 0 \\ 0 & -3.5 \end{bmatrix}, F_1 = \begin{bmatrix} -1 & 0.4 \\ 0 & -0.1 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} 0.2 & -0.8 \\ 0.4 & 0.5 \end{bmatrix}, \Upsilon = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$J_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, G_1 = [1 \quad 0], G_2 = [0.5 \quad 1], J_2 = 1, \varpi(\tau) = e^{-\tau}.$$

The activation function is selected as $\pi(s(\tau)) = [tanh(s_1(\tau)) \quad tanh(s_2(\tau))]^T$. Please note that $\pi(s(\tau))$ satisfies equation (3) and $L_\pi = 1$. In addition, the initial conditions for the neuron state and filter state are set to

$$s(x) = \begin{bmatrix} -3.2 \\ 1.5 \end{bmatrix}, \check{s}(x) = \begin{bmatrix} 2.4 \\ -2.1 \end{bmatrix}, x \in [-1, 0].$$

Meanwhile, assuming the time delay is given by $h(\tau) = 1 - 0.5exp(-\tau)$, and $h_1 = 0.5$, $h_2 = 1$. By setting $\rho = 0.2$, $\mu_1 = 0.3$ and $\mu_2 = 0.8$, we can obtain the optimal γ value of 0.88, and solve the LMI in equation (21) using a solver in MATLAB, which can generate

$$P_{11} = 10^3 * \begin{bmatrix} 0.9131 & 0.1235 \\ 0.1235 & 4.3826 \end{bmatrix}, M = \begin{bmatrix} 774.2565 \\ 110.1453 \end{bmatrix}.$$

Therefore, according to Theorem 2, the value of the matrix L can be calculated as:

$$L = P_{11}^{-1}M = \begin{bmatrix} 0.8501 \\ 0.0011 \end{bmatrix}$$

And, the systems (9) - (10) containing the gain matrix are the required quantized H_∞ filters.

The simulation results of the H_∞ filter design in this article are shown in Figure 1-4. Figure 1 shows the motion trajectory of state $s_1(\tau)$ and its estimated value $\check{s}_1(\tau)$; Similarly, the trajectory of state $s_2(\tau)$ and its estimated value $\check{s}_2(\tau)$ are shown in Figure 2; It can be observed that each estimated state closely follows its true counterpart, demonstrating the satisfactory tracking performance of the designed filter.

Figure 3 shows the dynamic variation of filtering error $\boldsymbol{\kappa(\tau)}$, and it is easy to conclude that the filtering error $\boldsymbol{\kappa(\tau)}$ ultimately converges to zero.

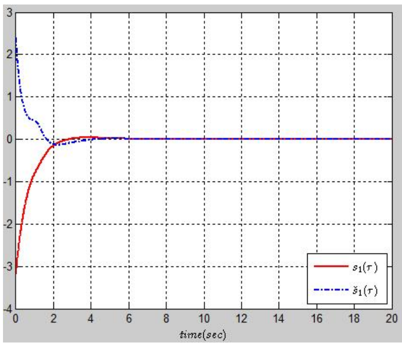

Figure 1. The Motion Trajectory of State $\boldsymbol{s_1(\tau)}$ and Its Estimate $\boldsymbol{\check{s}_1(\tau)}$

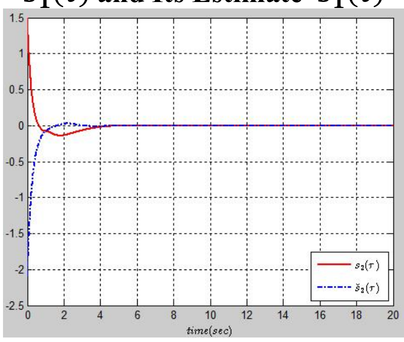

Figure 2. The Motion Trajectory of State $\boldsymbol{s_2(\tau)}$ and Its Estimate $\boldsymbol{\check{s}_2(\tau)}$

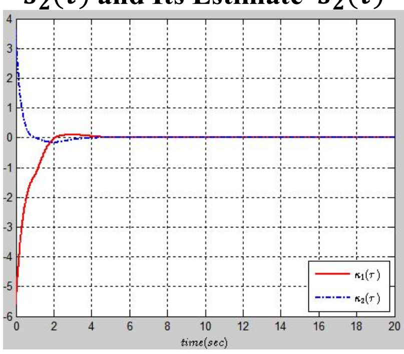

Figure 3. The Motion Trajectory of Filtering Error $\boldsymbol{\kappa(\tau)}$

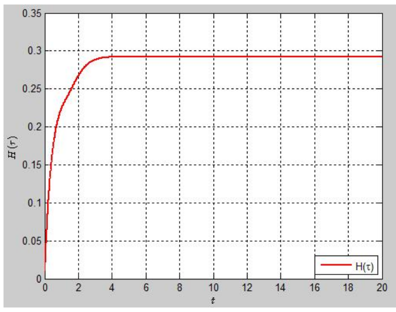

Figure 4. The Motion Trajectory of $\boldsymbol{H(\tau)}$

According to the simulation results shown in Figure 4, it can be easily observed that as the time parameter τ gradually increases, the performance indicator function $H(\tau)$ gradually stabilizes and eventually converges to a stable value of about 0.29. At the same time, the minimum H_∞ performance index pre-set in the design process is $\gamma_{min} = 0.88$. Obviously, the $H(\tau)$ value (0.29) at steady state is much smaller than $\gamma_{min} = 0.88$, indicating that the actual H_∞ noise attenuation level achieved by the system under the conditions of time-varying delay and output quantization is better than the performance bound required by the design. Therefore, this result fully verifies that the H_∞ filter designed in this paper can effectively suppress external disturbances and quantization errors, and has good robustness and expected filtering effect.

5. Conclusion

This article focuses on the H_∞ problem of neural network models with time-varying delays and output quantization, and draws the following conclusions:

(1) Stability and performance criteria: By constructing a new form of Lyapunov functional and combining it with the Bessel Legendre inequality, the conditions for ensuring asymptotic stability of the system and satisfying the given H_∞ performance index are derived. This criterion is expressed in the form of LMI and has low conservatism.

(2) Filter design method: Based on the above criteria, combined with decoupling strategies such as projection method and variable replacement method, as well as related inequality techniques, the solution of filter parameters is transformed into a solvable LMI constrained optimization problem, achieving effective design of the required H_∞ filter.

(3) Numerical verification: Through a numerical example, the filtering effect was compared and analyzed under different time delay change rates

and quantization accuracy. The results show that the designed filter can maintain system stability and meet the preset H_∞ performance under the influence of rapid time delay and quantization error, verifying the applicability and effectiveness of the method.

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