Longtime Dynamics of the Kirchhoff-Boussinesq-type Equation with Gentle Dissipation

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Abstract: The paper studies longtime dynamics of the Kirchhoff-Boussinesq-type equation with gentle dissipation, which is a class of nonlinear hyperbolic equations arising in nonlinear elasticity and dispersion wave propagation. The Kirchhoff-Boussinesq model combine the nonlocal Kirchhoff stress term with the Boussinesq dispersion, while gentle dissipation characterized by fractional damped operators adds subtle energy decay mechanisms. By giving full play to the dissipative effect of gentle damping, we shall establish the regularity of solutions of the K-B equation when the growth exponents of nonlinearities are up to the best critical index. Furthermore, we discuss the global attractor for Kirchhoff-Boussinesq equation of parabolic equation characters. Finally, we highlight open problems concerning upper semicontinuity, unbounded domain, non-autonomous dynamics for future research. This paper aims to understand the complex asymptotic behavior of Kirchhoff-Boussinesq equations with multi-physical effects.

Keywords: Kirchhoff-Boussinesq Equation; Gentle Dissipation; Energy Solution; Global Attractor

1. Introduction

The Kirchhoff-Boussinesq-type equation represents a fundamental class of equations that model the propagation of nonlinear dispersive waves in elastic media. Its origins trace back to Kirchhoff theory of vibrating strings and Boussinesq shallow water wave theory. Our main goal is to study the following Kirchhoff-Boussinesq-type equation in bounded domain $\Omega \subset R^N (N \geq 1)$,

$$\begin{aligned} &u_{tt} + \Delta^2 u + (-\Delta)^\alpha u_t - div\left\{\frac{\nabla u}{1+|\nabla u|^2}\right\} - \Delta g(u) = 0, \\ &\Delta u|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0, \\ &u_t(x,0) = u_1(x), u(x,0) = u_0(x), \end{aligned} \quad (1)$$

where $x \in \Omega$ with the smooth boundary $\partial\Omega$, Kirchhoff term $div\left\{\frac{\nabla u}{1+|\nabla u|^2}\right\}$ captures the dependence of wave speed on the total deformation energy, the dispersion term $\Delta^2 u$ accounts for frequency-dependent propagation, and the dissipation term $(-\Delta)^\alpha u_t$ means energy loss. The main results are concerned with the relationships among the growth exponent of $\Delta g(u)$ and the dissipative index $\alpha \in (0,1)$.

When $\alpha = 0$ in $(-\Delta)^\alpha u_t$, adding the damping coefficient $k > 0$, Chueshov and Lasiecka [1,2] studied 2D K–B

$$u_{tt} + ku_t + \Delta^2 u = div\left[g_0(\nabla u)\right] + \Delta g_1(u) - g_2(u), \quad (2)$$

The nonlinear feedback forces acting on the plate are represented by $g_0(\nabla u) = |\nabla u|^2 \nabla u$, $g_1 = u^2 + u$ and g_2 as demonstrated in [3], model (2) is derived as the limit of the Mindlin–Timoshenko system, which governs the dynamic behavior of plates with transverse shear deformations. For this two-dimensional model, the existence of a weak global attractor was established under the sufficient condition that the damping coefficient k is taken sufficiently large to suppress oscillation.

In fact, the difference of α can be realized in physics. The gentle dissipation $(-\Delta)^\alpha u_t (0 < \alpha < 1)$ in Eq (1) stronger than weak dissipation ($\alpha = 0$ in [4]) but weaker than the structural dissipation ($\alpha \in [1,2]$ in [5]). In this case $\alpha \in (0,1)$, what is the best critical exponent of the nonlinearity $g(u)$?

According to the idea in [6], since the Sobolev embedding $H_0^1 \hookrightarrow L^{q+1}$ imposes restrictions, the growth exponent of $g(u)$ in problem (1) $q^* \equiv \frac{N-2+4\alpha}{N-2}(N \geq 3)$ is the critical exponent. For dimension $N=1,2$, the critical exponent can be arbitrary. When $\alpha=1$, $q^*=\frac{N+2}{N-2}$ in [7], So longtime behaviors of weak solutionas as $N \geq 3$ are full of challenges.

When $1 \leq q < q^*$, we will obtain a bounded absorbing set with the higher global regularity and the quasi-stability in $X_{1-\alpha}$, then we devote to prove that the related dynamical system has a global attractor in natural energy space X_1. The results will further enrich the study of the K-B equation.

Nonlinear evolution equations with multi-physical effects have always been important research topics of the infinite-dimensional dynamical system since its beginning. The main results aims to overcome the technical difficulties caused by multi-physical effects.

Compared with Kirchhoff model (cf. [8]) and Boussinesq model (cf. [9]), the Kirchhoff-Boussinesq equation (1) poses significant challenges due to the interplay of several delicate mechanisms:

(i) the nonlocal nature of the Kirchhoff term leads to a loss of compactness in standard energy estimates; (ii) the dispersive regularization provided by the Boussinesq term improves regularity but complicates the spectral structure; (iii) the Kirchhoff term and $\Delta g(u)$ both occur. (iv) the critical $q=q^*$ or supercritical $q \geq q^*$ nonlinearities $\Delta g(u)$ that may induce finite-time blow-up in absence of sufficient strong damping; (v) gentle dissipation $(-\Delta)^{\alpha} u_t (0<\alpha<1)$ in problem (1) provides only weak restoring forces and makes the asymptotic analysis highly nontrivial. The questions subjected to the gentle dissipation are interesting both in mathematics and in physics.

Although the solution procedure remains similar to our earlier work [10], the physical model is modified. Relative to the previous model, the newly-added damping term changes the dynamic property of the system, which further leads to different qualitative results, the principal contribution of this paper is (i)We determine the critical exponent q^* of the nonlinearities $\Delta g(u)$ .For the range $1 \leq q < q^*$,the longtime behavior of Eq (1) exhibits parabolic-type characteristics. Importantly, we establish the existence of a bounded absorbing set that possesses full global regularity, as opposed to the partial regularity typically found in prior studies (see Theorems 2.1 and 3.1).(ii) Within the same parameter regime $1 \leq q < q^*$, we further construct the global attractor in X_1 (See Theorem 3.2).

The subsequent sections are structured as follows. The well-posedness of the solutions is examined in Section 2. The construction of the global attractor is carried out in Section 3. Section 4 concludes the paper and highlights some open questions for future research.

2. Well-Posedness

To simplify the presentation, we adopt the following notations throughout the paper.For $q \geq 1$, we denote $L^q = L^q(\Omega)$ .The norm $\|\cdot\|$ refers to the L^2 norm, while $\|\cdot\|_q$ denotes the L^q norm. The spaces $H^k = H^k(\Omega)$ are understood as the standard L^2 based Sobolev spaces, and H_0^k is defined as the closure of $C_0^k(\Omega)$ in H^k for $k>0$. The symbol $(\cdot,\cdot)$ is used both for the L^2-inner product and for duality pairings between dual spaces, while $C(\ldots)$ represents positive constants that depend on the parameters indicated in the parentheses.

The following operators and spaces are employed in our analysis. Setting $V_2 = H^2 \cap H_0^1$. We begin by introducing operator $A: V_2 \to V_2'$, Where V_2' denotes the dual space of V_2, $(Au, v) = (\Delta u, \Delta v)\ for\ any \quad u, v \in V_2.$

This operator is self-adjoint and strictly positive on V_2, which allows us to define its fractional powers A^s for $s \in R$. The resulting domains $V_s = D(A^{s/4})$ form Hilbert spaces endowed with the inner products and norms: $(u,v)_s = (A^{\frac{s}{4}}u, A^{\frac{s}{4}}v).$

A second operator $B: V_1 \to V_1'$ is defined through

$$\left(Bu,v\right)=\left(\frac{\nabla u}{\sqrt{1+\left|\nabla u\right|^{2}}},\nabla v\right) for\ any \quad u,v\in V_{1}.$$

with V_1' denoting the dual of V_1. We set $X_s = V_s \times V_{-2+s}, s \in R.$

In particular, the case $s=1$ corresponds to the energy space $X_1 = V_1 \times V_{-1}$, while $s=1-\alpha$ gives $X_{1-\alpha} = V_{1-\alpha} \times V_{-1-\alpha}$. For $\alpha \in (0,1)$, $0<1-\alpha<1$.

Each of these spaces X_s is furnished with the usual graph norm, taking X_1 as a representative example, we have $\left\|(u,u_t)\right\|_{X_1}^2 = \|u\|_{V_1}^2 + \|u_t\|_{V_{-1}}^2$. It is straightforward to verify that all spaces X_s are Hilbert spaces.

Turning to the evolution equation, we recast (1) into an operator-valued formulation. Upon applying the fractional power $A^{-1/2}$ to the resultant expression, we arrive at the following transformed equation:

$$\begin{aligned}&A^{-\frac{1}{2}}u_{tt}+A^{\frac{1}{2}}u+A^{\frac{-1+\alpha}{2}}u_t+A^{-\frac{1}{2}}Bu+g\left(u\right)=0,\\&u\left(0\right)=u_0,u_t\left(0\right)=u_1.\end{aligned} \tag{3}$$

Theorem 2.1

Suppose that the following assumptions hold

(1) $g\in C'(R)$ and $g(0)=0$,

the derivative g' is subject to the growth constraint

$$\left|g'(s)\right|\le C(1+\left|s\right|^{q-1}), s\in R, \tag{4}$$

Where the exponent q lies in the range $q\in\left[1,q^*\right)$, with $q^*\equiv\frac{N-2+4\alpha}{N-2}(N\ge 3)$.

Additionally, the asymptotic condition $g'(s)>-\sqrt{\lambda_1}+\sigma$ is imposed, where σ is sufficiently large and $\lambda_1>0$ stands for the first eigenvalue of the operator A. (2) $(u_0,u_1)\in X_1,$ with $\|u_0\|_{V_1}^2+\|u_1\|_{V_{-1}}^2\le R.$ Then problem (1) admits an energy solution u, $(u,u_t)\in C\left(R^+,X_1\right)$, and the boudedness estimate of $\varphi_u=(u,u_t)$

$$\left\|\varphi_u(t)\right\|_{X_1}^2+\int_t^\infty\left\|u_t(\tau)\right\|_{V_{-1+\alpha}}^2 d\tau\le C(R), t\ge 0. \tag{5}$$

For any $b>0$, $(u_t,u_{tt})\in L^\infty\left(b,T;X_{1-\alpha}\right)\cap L^2\left(b,T;X_1\right)$.

Furthermore, the following energy estimate is obtained for all $t>0$ and sufficiently small constant $\varepsilon>0$

$$\begin{aligned}&\left\|u_t(t)\right\|_{V_{1-\alpha}}^2+\left\|u_{tt}(t)\right\|_{V_{-1-\alpha}}^2+\int_t^{t+1}\left(\left\|u_{tt}(\tau)\right\|_{V_{-1}}^2+\left\|u_t(\tau)\right\|_{V_1}^2\right)d\tau\\&\le\frac{t^{\frac{2}{\alpha}}+1}{t^{\frac{2}{\alpha}}}C(R)e^{-\varepsilon t}+C.\end{aligned} \tag{6}$$

Proof of Theorem 2.1

The proof is initiated by deriving a series of a priori estimates for the solutions of (1). To this aim, we multiply Eq. (3) by u_t,

$$\left(A^{-\frac{1}{2}}u_{tt},u_t\right)+\left(A^{\frac{1}{2}}u,u_t\right)+\left(A^{\frac{-1+\alpha}{2}}u_t,u_t\right)=\left(-A^{-\frac{1}{2}}Bu,u_t\right)+\left(-g(u),u_t\right)$$

leading to

$$\frac{d}{dt}E_1(\varphi_u)+\left\|A^{\frac{-1+\alpha}{4}}u_t\right\|^2+\left(A^{-\frac{1}{2}}Bu,u_t\right)=0, \tag{7}$$

where

$$\begin{aligned}E_1(\varphi_u)&=\frac{1}{2}\left\|A^{-\frac{1}{4}}u_t\right\|^2+\frac{1}{2}\left\|A^{\frac{1}{4}}u\right\|^2+\int_\Omega\int_0^u g(\tau)d\tau dx\\&\ge\frac{1}{4}\left(\left\|A^{-\frac{1}{4}}u_t\right\|^2+\left(1-\frac{\varepsilon}{\sqrt{\lambda_1}}\right)\left\|A^{\frac{1}{4}}u\right\|^2\right)-C,\\&0<\sqrt{\lambda_1}-\varepsilon\ll 1,\end{aligned}$$

$$\begin{aligned}\left|\left(A^{-\frac{1}{2}}Bu,u_t\right)\right|&\le\|Bu\|_{V_{-1}}\left\|A^{-\frac{1}{2}}u_t\right\|_{V_1}\\&\le\frac{1}{4}\left\|A^{-\frac{1}{4}}u_t\right\|^2+C\end{aligned}$$

and

Integrating (7) over the time interval gives (5). Then, combining (3) with (5), we deduce

$$\left\|u_{tt}(t)\right\|_{V_{-3}}\le C(R), t\ge 0. \tag{8}$$

The time differentiation of (3) shows that $w=u_t$ satisfies

$$\begin{aligned}&A^{-\frac{1}{2}}w_{tt}+A^{\frac{1}{2}}w+A^{\frac{\alpha-1}{2}}w_t+g'(u)w\\&=A^{-\frac{1}{2}}M(u,w).\end{aligned} \tag{9}$$

where

$$M(u,w)=div\left\{\frac{\nabla w}{\sqrt{1+\left|\nabla u\right|^2}}-\frac{\nabla u\cdot\nabla w\cdot\nabla u}{\left(1+\left|\nabla u\right|^2\right)^{\frac{3}{2}}}\right\}.$$

With $A^{-\frac{\alpha}{2}}w_t+\varepsilon w$ chosen as the multiplier in (9), we derive,

$$\frac{d}{dt}E_2(w,w_t)+\varepsilon\left\|A^{\frac{1}{4}}w\right\|^2+(1-\varepsilon)\left\|A^{-\frac{1}{4}}w_t\right\|^2$$
$$=-\left(g'(u)w,A^{-\frac{\alpha}{2}}w_t+\varepsilon w\right)$$
$$+\left(A^{-\frac{1}{2}}M(u,w),A^{-\frac{\alpha}{2}}w_t+\varepsilon w\right), \tag{10}$$

where

$$E_2(w,w_t)=\frac{1}{2}\left[\left\|A^{-\frac{1+\alpha}{4}}w_t\right\|^2+\left\|A^{\frac{1-\alpha}{4}}w\right\|^2\right]$$
$$+\varepsilon\left(\left\|A^{\frac{\alpha-1}{4}}w\right\|^2+\left(A^{-\frac{1}{4}}w,A^{-\frac{1}{4}}w_t\right)\right)$$
$$\sim\|w\|_{V_{1-\alpha}}^2+\|w_t\|_{V_{-1-\alpha}}^2,$$

We now examine the inhomogeneous term in (10).

$$\left|\left(A^{-\frac{1}{2}}M(u,w),A^{-\frac{\alpha}{2}}w_t+\varepsilon w\right)\right|$$
$$=\left|\left(div\left\{\frac{\nabla w}{\sqrt{1+|\nabla u|^2}}-\frac{\nabla u\cdot\nabla w\cdot\nabla u}{\left(1+|\nabla u|^2\right)^{\frac{3}{2}}}\right\},A^{-\frac{\alpha+1}{2}}w_t+\varepsilon A^{-\frac{1}{2}}w\right)\right|$$
$$\le C\left\|\frac{\nabla w}{\sqrt{1+|\nabla u|^2}}-\frac{\nabla u\cdot\nabla w\cdot\nabla u}{\left(1+|\nabla u|^2\right)^{\frac{3}{2}}}\right\|\left\|\nabla\left(A^{-\frac{\alpha+1}{2}}w_t+\varepsilon A^{-\frac{1}{2}}w\right)\right\|$$
$$\le\frac{\varepsilon}{4}\|w\|_{V_1}^2+\frac{1}{4}\|w_t\|_{V_{-1}}^2+C(\varepsilon)\|w\|_{V_{-1}}^2 \tag{11}$$

Combined with (3), An application of the interpolation theorem

$$\left(g'(u)w,A^{-\frac{\alpha}{2}}w_t+\varepsilon w\right)$$
$$\le C\int_\Omega\left(1+|u|^{q-1}\right)|w|\left(\left|A^{-\frac{\alpha}{2}}w_t\right|+\varepsilon|w|\right)dx$$
$$\le C\varepsilon\left(1+\|u\|_{q+1}^{q-1}\right)\|w\|_{q+1}^2$$
$$+C\left(1+\|u\|_{\frac{N(q-1)}{2\alpha-\delta}}^{p-1}\right)\|w\|_{\frac{2N}{N-2(1-\delta)}}\left\|A^{-\frac{\alpha}{2}}w_t\right\|_{\frac{2N}{N-2(2\alpha-1)}}$$
$$\le C\varepsilon\|w\|_{V_{1-\delta}}^2+C\left(1+\|u\|_{\frac{2N}{N-2}}^{q-1}\right)\|w\|_{V_{1-\delta}}\|w_t\|_{V_{-1}}$$
$$\le\frac{\varepsilon}{4}\|w\|_{V_1}^2+\frac{1}{4}\|w_t\|_{V_{-1}}^2+C(\varepsilon)\|w\|_{V_{-1}}^2, \tag{12}$$

where $\frac{N(q-1)}{2\alpha-\delta}\le\frac{2N}{N-2}$ for $0<\delta<<1$.

Inserting (11)-(12) into (10), we get

$$\frac{d}{dt}E_2(w,w_t)+\frac{\varepsilon}{2}E_2(w,w_t)+\frac{\varepsilon}{4}\left(\|w\|_{V_1}^2+\|w_t\|_{V_{-1}}^2\right)$$
$$\le C(R)\|w\|_{V_{-1}}^2. \tag{13}$$

We first multiply (13) by $t^{\frac{2}{\alpha}}$ ($\frac{2}{\alpha}>2$)in the regime $0<t\le1$ and $0<\alpha<1$. This yields

$$\frac{d}{dt}\left(t^{\frac{2}{\alpha}}E_2(w,w_t)\right)+\frac{\varepsilon}{2}t^{\frac{2}{\alpha}}E_2(w,w_t)$$
$$+\frac{\varepsilon}{4}t^{\frac{2}{\alpha}}\left(\|w\|_{V_1}^2+\|w_t\|_{V_{-1}}^2\right)$$
$$\le C(R)t^{\frac{2}{\alpha}}\|w\|_{V_{-1}}^2+\frac{2}{\alpha}t^{\frac{2}{\alpha}-1}\frac{d}{dt}E_2(w,w_t), \tag{14}$$

where $t^{\frac{2}{\alpha}}\|w\|_{V_{-1}}^2\le\|w\|_{V_{-1}}^2$ for $0<t^{\frac{2}{\alpha}}\le1$.

Applying the Gagliardo–Nirenberg interpolation inequality,

$$t^{\frac{2}{\alpha}-1}\|w_t\|_{V_{-1-\alpha}}^2\le Ct^{\frac{2}{\alpha}-1}\|w_t\|_{V_{-1}}^{2-\alpha}\|w_t\|_{V_{-3}}^{\alpha}$$
$$\le\frac{\varepsilon}{8}t^{\frac{2}{\alpha}}\|w_t\|_{V_{-1}}^2+C\|w_t\|_{V_{-3}}^2, \tag{15}$$

$$t^{\frac{2}{\alpha}-1}\|w\|_{V_{1-\alpha}}^2\le Ct^{\frac{2}{\alpha}-1}\|w\|_{V_{-1}}^{2-\alpha}\|w\|_{V_{-1}}^{\alpha}$$
$$\le\frac{\varepsilon}{8}t^{\frac{2}{\alpha}}\|w\|_{V_1}^2+C\|w\|_{V_{-1}}^2. \tag{16}$$

Let $\beta=\varepsilon/8$, inserting (15)-(16) into (14),we can get that

$$\frac{d}{dt}\left(t^{\frac{2}{\alpha}}E_2(w,w_t)\right)+\beta t^{\frac{2}{\alpha}}E_2(w,w_t)$$
$$+\beta t^{\frac{2}{\alpha}}\left(\|w\|_{V_1}^2+\|w_t\|_{V_{-1}}^2\right)\le C\left(\|w\|_{V_{-1}}^2+\|w_t\|_{V_{-3}}^2\right). \tag{17}$$

With the Gronwall lemma applied to (17), we derive

$$\|u_t(t)\|_{V_{1-\alpha}}^2+\|u_{tt}(t)\|_{V_{-1-\alpha}}^2$$
$$+\int_t^{t+1}\left(\|u_t(\tau)\|_{V_1}^2+\|u_{tt}(\tau)\|_{V_{-1}}^2\right)d\tau$$
$$\le\frac{1}{t^{\frac{2}{\alpha}}}\left(C(R)e^{-\beta t}+C\right),\qquad 0<t\le1. \tag{18}$$

In the regime $t>1$,

Gronwall inequality applied to (13) gives

$$\|u_t(t)\|_{V_{1-\alpha}}^2+\|u_{tt}(t)\|_{V_{-1-\alpha}}^2$$
$$+\int_t^{t+1}\left(\|u_t(\tau)\|_{V_1}^2+\|u_{tt}(\tau)\|_{V_{-1}}^2\right)d\tau$$
$$\le C(R)e^{-\beta t}+C. \tag{19}$$

We combine $0<t\le1$ in (18) and $t>1$ in (19) gives inequality (6).

We now proceed with the Galerkin method by seeking approximate solutions u^n of (3) in the N-dimensional space spanned by the eigenfunctions $\{h_1,h_2\cdots h_n\}$, $u^n(t)=\sum_{j=1}^{n}T_{jn}(t)h_j$, here $\{h_j\}$ is is the eigenbasis of A satisfying $Ah_j = \lambda_j h_j, j = 1,2,\cdots$ .The coefficients are determined by $T_{jn}(t) = (u^n, h_j)$,

$$(A^{-\frac{1}{2}}u_{tt}^n, h_j) + (A^{\frac{1}{2}}u^n, h_j) + (A^{\frac{\alpha-1}{2}}u_t^n, h_j) +$$
$$(A^{-\frac{1}{2}}Bu^n, h_j) + (g(u^n), h_j) = 0, t > 0,$$
$$(u^n(0), u_t^n(0)) = (u_{0n}, u_{1n}) \to (u_0, u_1) \, in \quad X_1 . \tag{20}$$

Obviously, the estimates (5)-(6) hold for u^n, and

$$\|g(u^n)\|_{1+1/q} \le C\left(1 + \|u^n\|_{q+1}^q\right) \le C.$$

So

$$(u^n, u_t^n) \to (u, u_t) \quad weakly^* \, in \quad L_{loc}^\infty(R^+, X_1),$$
$$u_{tt}^n \to u_{tt} \quad in \quad L_{loc}^\infty(R^+, V_{-3}).$$

Applying limit Lemma to (20), we have

$$u^n \to u \, in \quad C([0,T]; V_{1-\alpha}) \, a.e. \quad in \quad Q_T = \Omega \times [0,T]$$
$$u_t^n \to u_t \, in \quad L^2([0,T]; V_{-1+\alpha}).$$

When $1 \le q < q^*$, for any $b > 0$

$$(u_t^n, u_{tt}^n) \to (u_t, u_{tt}) \, weakly^* \quad in \quad L^\infty(b, T; X_{1-\alpha}),$$

Hence

$$g(u^n) \to g(u) \, weakly^* \quad in \quad L^\infty\left(0, T, L_{1+1/q}\right),$$
$$A^{-\frac{1}{2}}Bu^n \to A^{-\frac{1}{2}}Bu \, weakly^* \quad in \quad L^\infty(0, T, V_{-1}).$$

Equation (20) is passed to the limit as $n \to \infty$, from which the limiting function u is obtained as an energy solution to problem (1), $\varphi_u = (u, u_t) \in L^\infty(R^+, X_1) \cap C_w(R^+, X_1)$.

This establishes Theorem 2.1.

Theorem 2.2

Under the hypotheses of Theorem 2.1， the solution to problem (1) is Lipschitz continuous，

$$\|(z, z_t)(t)\|_{X_1}^2 \le C(R,T)\|(z, z_t)(0)\|_{X_1}^2 , \tag{21}$$

Specifically, for any $0 \le t \le T$, the solution u possesses quasi-stability in the weaker space $X_{1-\alpha}$,

$$\|(z, z_t)(t)\|_{X_{1-\alpha}}^2 \le C(R)e^{-\varepsilon t}\|(z, z_t)(0)\|_{X_{1-\alpha}}^2$$
$$+\int_0^t e^{-\varepsilon(t-\tau)}\|z(\tau)\|_{V_{-1}}^2 \, d\tau, \tag{22}$$

where $z = u - v$, with u and v being the limiting solutions of (3) arising from the initial pairs (u_0, u_1) and (v_0, v_1), respectively.

Proof of Theorem 2.2

z solves

$$A^{-\frac{1}{2}}z_{tt} + A^{\frac{1}{2}}z + A^{\frac{\alpha-1}{2}}z_t$$
$$= -\left[A^{-\frac{1}{2}}(Bu - Bv) + g(u) - g(v)\right].$$
$$z(0) = u_0 - v_0 \equiv z_0, z_t(0) = u_1 - v_1 \equiv z_1. \tag{23}$$

By testing Eq. (23) against $A^{-\frac{\alpha}{2}}z_t + \varepsilon z$, we arrive at

$$\frac{d}{dt}E_3(z, z_t) + \varepsilon\left\|A^{\frac{1}{4}}z\right\|^2 + (1-\varepsilon)\left\|A^{-\frac{1}{4}}z_t\right\|^2$$
$$= -\left(g(u) - g(v), A^{-\frac{\alpha}{2}}z_t + \varepsilon z\right)$$
$$-\left(A^{-\frac{1}{2}}(Bu - Bv), A^{-\frac{\alpha}{2}}z_t + \varepsilon z\right), \tag{24}$$

where

$$E_3(z, z_t) = \frac{1}{2}\left[\left\|A^{-\frac{1+\alpha}{4}}z_t\right\|^2 + \left\|A^{\frac{1-\alpha}{4}}z\right\|^2\right]$$
$$+\varepsilon\left(\left\|A^{\frac{\alpha-1}{4}}z\right\|^2 + \left(A^{-\frac{1}{4}}z, A^{-\frac{1}{4}}z_t\right)\right)$$
$$\sim \|z\|_{V_{1-\alpha}}^2 + \|z_t\|_{V_{-1-\alpha}}^2 ,$$

Choosing a sufficiently small parameter $\varepsilon > 0$ and proceeding as in (11)–(12), we derive

$$\left|\left(A^{-\frac{1}{2}}(Bu - Bv), A^{-\frac{\alpha}{2}}z_t + \varepsilon z\right)\right|$$
$$\le \left|\left(\int_0^1 [\frac{\nabla z}{\sqrt{1+|\nabla z_\theta|^2}} - \frac{\nabla z_\theta \cdot \nabla z \cdot \nabla z_\theta}{\left(1+|\nabla z_\theta|^2\right)^{\frac{3}{2}}}]d\theta, A^{-\frac{1}{2}}\left(A^{-\frac{\alpha+1}{2}}z_t + \varepsilon A^{-\frac{1}{2}}z\right)\right)\right|$$
$$\le \frac{\varepsilon}{4}\|z\|_{V_1}^2 + \frac{1}{4}\|z_t\|_{V_{-1}}^2 + C(\varepsilon)\|z\|_{V_{-1}}^2 , \tag{25}$$

where $z_\theta = \theta u + (1-\theta)v$ for $0 < \theta < 1$. And,

$$\left|\left(g(u) - g(v), A^{-\frac{\alpha}{2}}z_t + \varepsilon z\right)\right|$$
$$\le C\int_\Omega \left(1 + |u|^{p-1} + |v|^{p-1}\right)|z|\left(\left|A^{-\frac{\alpha}{2}}z_t\right| + \varepsilon|z|\right)dx$$
$$\le \frac{\varepsilon}{4}\|z\|_{V_1}^2 + \frac{1}{4}\|z_t\|_{V_{-1}}^2 + C(\varepsilon)\|z\|_{V_{-1}}^2 . \tag{26}$$

Substituting (25) and (26) into (24), we obtain the differential inequality

$$\frac{d}{dt}E_3(z, z_t) + \frac{\varepsilon}{2}E_3(z, z_t) + \frac{\varepsilon}{4}\left(\|z\|_{V_1}^2 + \|z_t\|_{V_{-1}}^2\right)$$
$$\le C\|z\|_{V_{-1}}^2 , \tag{27}$$

Applying Gronwall's lemma to (27), we derive the desired estimates (21) and (22).

3. Existence of Global Attractor

We begin by defining the map $S(t): X_1 \to X_1$,

$S(t)\varphi_0 = \varphi(t) = (u(t), u_t(t)), \quad t \geq 0,$

where $\varphi_0 \in X_1$ and u is the energy solution of (1) with $\varphi(0) = \varphi_0$. Theorem 2.2 establishes the uniqueness of u and shows that $S(t)$ is a semigroup on X_1 that is Lipschitz continuous in the $X_{1-\alpha}$-topology.

From the perspective of dynamical systems, a central question is the existence of global attractor, which is intimately linked to the balance between nonlinear energy input and dissipative losses. In the gentle dissipation regime, the damping may be so weak that requiring refined methods such as asymptotically compact techniques.

The existence of a global attractor relies on two sufficient conditions: (i) the asymptotic compactness of the semigroup $S(t)$; (ii) the existence of a bounded absorbing set $\Theta_R \subset X_1$ that attracts every bounded subset of X_1.

Theorem 3.1

(Higher-regularity bounded absorbing set) Suppose that the assumptions of Theorem 2.1 hold, the semigroup $S(t)$ is shown to be dissipative in X_1 and to have a forward invariant Θ_R that is bounded in $V_{1+\alpha} \times V_{-1+\alpha}$.

Proof of Theorem 3.1

(i) Construction of the bounded absorbing set. We begin by taking u as a multiplier in Eq. (3), which gives

$$\frac{d}{dt}\left[\left(A^{\frac{1}{4}}u,\ A^{\frac{1}{4}}u_t\right) + \frac{1}{2}\left\|A^{\frac{\alpha-1}{4}}u\right\|^2\right] + \left\|A^{\frac{1}{4}}u\right\|^2 + \left(A^{-\frac{1}{2}}Bu, u\right) + \left(g(u), u\right) = \left\|A^{-\frac{1}{4}}u_t\right\|^2, \tag{28}$$

Adding ε times (28) to (7), we obtain

$$\frac{d}{dt}E_4(\varphi_u) + D(\varphi_u) = 0, \tag{29}$$

Where

$$\begin{aligned} E_4(\varphi_u) &= E_1(\varphi_u) + \varepsilon\left[\left(A^{\frac{1}{4}}u,\ A^{\frac{1}{4}}u_t\right) + \frac{1}{2}\left\|A^{\frac{\alpha-1}{4}}u\right\|^2\right] \\ &= \frac{1}{2}\left\|A^{-\frac{1}{4}}u_t\right\|^2 + \frac{1}{2}\left\|A^{\frac{1}{4}}u\right\|^2 + \int_\Omega\int_0^u g(\tau)d\tau dx \\ &\quad + \varepsilon\left[\left(A^{\frac{1}{4}}u,\ A^{\frac{1}{4}}u_t\right) + \frac{1}{2}\left\|A^{\frac{\alpha-1}{4}}u\right\|^2\right] \\ &\geq \frac{1}{2}\left(\left\|A^{-\frac{1}{4}}u_t\right\|^2 + \eta\left\|A^{\frac{1}{4}}u\right\|^2\right) - C, \end{aligned}$$

$$\begin{aligned} D(\varphi_u) &= \left\|A^{\frac{\alpha-1}{4}}u_t\right\|^2 - \varepsilon\left\|A^{-\frac{1}{4}}u_t\right\|^2 + \left(A^{-\frac{1}{2}}Bu, u_t\right) \\ &\quad + \varepsilon\left[\left\|A^{\frac{1}{4}}u\right\|^2 + \left(A^{-\frac{1}{2}}Bu, u\right) + \left(g(u), u\right)\right] \\ &\geq \eta\varepsilon\left(\left\|A^{\frac{1}{4}}u\right\|^2 + \left\|A^{-\frac{1}{4}}u_t\right\|^2\right) - C\varepsilon \end{aligned} \tag{30}$$

for $\eta > 0$ stands for a small constant.

Substituting the estimate (30) into the differential inequality (29),

$$\frac{d}{dt}E_4(\varphi_u) + \eta\varepsilon\|\varphi_u\|_{X_1}^2 \leq C\varepsilon. \tag{31}$$

This yields the desired decay estimate

$\sup_{t \in R^+} E_4(\varphi_u(t)) \geq -C,$

$E_4(\varphi_u(0)) \leq C\left(\|\varphi_u(0)\|_X^2\right).$

Taking $\delta = \eta\varepsilon,\ \gamma = \varepsilon$ to (31), we arrive at

$E_4(\varphi_u) \leq \sup_{\xi \in X_1}\left\{E_4(\xi) : \|\xi\|_{X_1}^2 \leq \frac{C+1}{\eta}\right\} \triangleq K_1,$

for $t \geq t_A \equiv \frac{C+1}{\eta}.$

Therefore,

$$\|\varphi_u(t)\|_{X_1}^2 \leq K_1 + C \equiv R, t \geq t_A. \tag{32}$$

We deduce from (32) that, for a suitably chosen R, $B_R = \left\{\xi \in X_1 \middle| \|\xi\|_{X_1} \leq R\right\}$ is an absorbing set of the operator $S(t)$. Therefore, $(S(t), X_1)$ *is* dissipative.

(ii)(Bounded absorbing set with enhanced regularity) The estimate (6) in Theorem 2.1 shows that the absorbing set B_R has the higher partially regularity provided that $u_t(t) \in V_{1-\alpha}$.

Let $(u_0, u_1) \in B_R$, multiplying Eq. (3) by $A^{\frac{1}{2}}u$, we have

$$\frac{d}{dt}\left[(u, u_t) + \frac{1}{2}\left\|A^{\frac{\alpha}{4}}u\right\|^2\right] + \left\|A^{\frac{1}{2}}u\right\|^2 + \int_\Omega \frac{|\nabla u|^2}{\sqrt{1+|\nabla u|^2}}dx + \int_\Omega g'(u)|\nabla u|^2 dx = \|u_t\|^2. \tag{33}$$

Integrating (33) over $(t, t+1)$,

$$\int_t^{t+1}\left\|A^{\frac{1}{2}}u(\tau)\right\|^2 d\tau \leq C\left(\left\|A^{\frac{\alpha}{4}}u(t)\right\|^2 + \left\|A^{-\frac{\alpha}{4}}u_t(t)\right\|^2\right) + \int_t^{t+1}\|u_t(\tau)\|^2 d\tau + \int_t^{t+1}\left\|A^{\frac{1}{4}}u(\tau)\right\|^2 d\tau \leq C(R), t \geq b.$$

By the estimate (6),

$\int_t^{t+1}\|u_{tt}(\tau)\|_{V_{-1}}^2 d\tau \le C(R), t \ge b.$

Compared with $A^{\frac{1}{2}}u$,the current convergence order has not yet been improved to the desred level, testing Eq. (3) by Au ,

$$\frac{1}{2}\frac{d}{dt}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2+\left\|A^{\frac{3}{4}}u\right\|^2+\left(Bu, A^{\frac{1}{2}}u\right)+\left(g(u), Au\right)$$

$$=\left(-u_{tt}, A^{\frac{1}{2}}u\right)\le\frac{1}{4}\left\|A^{\frac{3}{4}}u\right\|^2+C\|u_{tt}\|_{V_{-1}}^2,$$

Where

$$\left|\left(g(u), Au\right)\right|\le C\int_\Omega\left(1+|u|^{q-1}\right)|\nabla u|\left|A^{\frac{3}{4}}u\right|dx$$

$$\le C\|\nabla u\|\left\|A^{\frac{3}{4}}u\right\|+C\left\||u|^{q-1}|\nabla u|\right\|\left\|A^{\frac{3}{4}}u\right\|$$

$$\le\frac{1}{4}\left\|A^{\frac{3}{4}}u\right\|^2+C\|u\|_{V_1}^2+C\left\||u|^{q-1}|\nabla u|\right\|^2$$

$$\le\frac{1}{4}\left\|A^{\frac{3}{4}}u\right\|^2+C\|u\|_{V_1}^2+C\|u\|_{V_1}^{2\theta_0}+\left\|A^{\frac{1}{2}}u\right\|^4$$

$$\left|\left(Bu, A^{\frac{1}{2}}u\right)\right|$$

$$=\int_\Omega\left[\frac{|\Delta u|^2}{\sqrt{1+|\nabla u|^2}}-\frac{|\Delta u\cdot\nabla u|^2}{\left(1+|\nabla u|^2\right)^{\frac{3}{2}}}\right]dx\ge 0,$$

where $\theta_0=\frac{N}{2N-(N-4)(q+1)}$ which implies $(q-1)\theta_0\le 1$.

We have

$$\frac{d}{dt}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2+\eta\left\|A^{\frac{3}{4}}u\right\|^2\le C\|u\|_{V_1}^2+C\|u\|_{V_1}^{2\theta_0}+\left\|A^{\frac{1}{2}}u\right\|^4+\|u_{tt}\|_{V_{-1}}^2, \tag{34}$$

When $\frac{b}{2}<t\le\frac{b}{2}+1$, multiplying (34) by $\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}$,

$$\frac{d}{dt}\left(\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2\right)+\eta\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{3}{4}}u\right\|^2$$

$$\le C\left(\|u\|_{V_1}^2+\|u\|_{V_1}^{2\theta_0}+\left\|A^{\frac{1}{2}}u\right\|^4+\|u_{tt}\|_{V_{-1}}^2\right)$$

$$+\frac{1}{1-\alpha}\left(t-\frac{b}{2}\right)^{\frac{\alpha}{1-\alpha}}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2.$$

Since V_2 is embedded into $V_{1+\alpha}$, $V_{1+\alpha}$ is embedded into V_1, and $\left(t-\frac{b}{2}\right)^{\frac{2}{1-\alpha}}\le\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}$ for $\frac{b}{2}<t\le\frac{b}{2}+1$,

$$\left(t-\frac{b}{2}\right)^{\frac{\alpha}{1-\alpha}}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2$$

$$\le C\left\|A^{\frac{1}{4}}u\right\|^{2-\alpha}\left(t-\frac{b}{2}\right)^{\frac{\alpha}{1-\alpha}}\left\|A^{\frac{3}{4}}u\right\|^{\alpha}$$

$$\le\varepsilon\left(t-\frac{b}{2}\right)^{\frac{2}{1-\alpha}}\left\|A^{\frac{3}{4}}u\right\|^2+C(\varepsilon)\left\|A^{\frac{1}{4}}u\right\|^2$$

$$\le\varepsilon\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{3}{4}}u\right\|^2+C(\varepsilon)\left\|A^{\frac{1}{4}}u\right\|^2.$$

Then

$$\frac{d}{dt}\left(\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2\right)+\eta\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2$$

$$+\eta\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{3}{4}}u\right\|^2$$

$$\le C\left(\|u\|_{V_1}^2+\|u\|_{V_1}^{2\theta_0}+\|u\|_{V_2}^4+\|u_{tt}\|_{V_{-1}}^2\right). \tag{35}$$

Making use of (35) and (6), we have

$$\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}\left\|A^{\frac{\alpha+1}{4}}u\right\|^2\le C(\mathrm{R},b),$$

$$\|u\|_{V_{1+\alpha}}^2\le\frac{1}{\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}}C(\mathrm{R},\mathrm{b}), t\in\left(\frac{b}{2},\frac{b}{2}+1\right]. \tag{36}$$

When $t>\frac{b}{2}+1$, using (34) over $\left(\frac{b}{2}+1,\ t\right)$, we obtain

$$\|u\|_{V_{1+\alpha}}^2\le C(\mathrm{R},\mathrm{b}),\qquad t>\frac{b}{2}+1. \tag{37}$$

The combination (36) and (37), we get

$$\|u\|_{V_{1+\alpha}}^2\le\left(1+\frac{1}{\left(t-\frac{b}{2}\right)^{\frac{1}{1-\alpha}}}\right)C(\mathrm{R},\mathrm{b}), t>\frac{b}{2}. \tag{38}$$

Chosing $t_B>0$ such that $T(t)B_R\subset B_R$ for $t\ge t_B$. Let

$$\Theta_R=\left[\bigcup_{t\ge t_B+1}T(t)B_R\right]_{X_{1-\alpha}}, \tag{39}$$

which is closed in $X_{1-\alpha}$ for $[\ ]_{X_{1-\alpha}}$ stands for the closure in $X_{1-\alpha}$. By the estimates (6) in Theorem 2.1.(ii) and (38), on Θ_R,

$\|\varphi(t)\|_{X_{1+\alpha}}^2\le\|u(t)\|_{V_{1+\alpha}}^2+\|u(t)\|_{V_{1-\alpha}}^2\le C(R)$

where $V_{1-\alpha}$ is embedded into $V_{-1+\alpha}, \alpha\in(0,1)$.

Here Θ_R ($S(t)\Theta_R\subset\Theta_R, t\ge 0$) is bounded in the

more regular space $X_{1+\alpha} = V_{1+\alpha} \times V_{-1+\alpha}$, which is compactly embedded in X_1.
We obtain the desired result of Theorem 3.1.
Theorem 3.2. Under the hypotheses of Theorem 2.1 with $1 \le q < q^*$, $S(t)$ admits a global attractor in X_1.
Proof of Theorem 3.2.
It suffices to verify the two standard criteria for the existence of a global attractor. The dissipativity of $S(t)$. For each bounded subset $B \in X_1$, there exists a time $t_B > 0$, such that, for every $t \ge t_B$, $S(t)B_R \subset B_R$ holds. On account of Θ_R is bounded in $V_{1+\alpha} \times V_{-1+\alpha}$ and $V_{1+\alpha} \times V_{-1+\alpha}$ is embedded into X_1, we obtain $S(t)$ is uniformly compact. Therefore, the system $(S(t), X_1)$ is dissipative and uniformly compact, we know that the system $(S(t), X_1)$ has global attractor.

4. Conclusions and Open Problems

In this paper, we devote to study Kirchhoff-Boussinesq type equations, describing the dynamical behavior of the plates. Focusing on typical model of Kirchhoff nonlinearity and the critical growth of Boussinesq nonlinearity, we explore the multi-physical effects of Kirchhoff-Boussinesq equation. Kirchhoff nonlinearity and Boussinesq nonlinearity exist simultaneously, so that the dissipation effect and the energy concentration effect caused by multiple nonlinearities happen at the same time, which makes this research complex and challenging.
Our research is carried out under the assumption that the divergence operator commutes with the fractional square root $A^{\frac{1}{2}}$ of the positive definite operator. This commutativity holds under $\Delta u|_{\partial\Omega} = 0, u|_{\partial\Omega} = 0$, whereas it fails to hold under $u|_{\partial\Omega} = 0, \frac{\partial u}{\partial n}|_{\partial\Omega} = 0$.
We conclude by listing a number of open problems that we believe are central to future research.

Open Problem 1 (global attractors A_α with upper semicontinuity)
For the exponent α, define $S_\alpha(t): X_1 \to X_1$ for every $\alpha \in (0,1)$,

$$S_\alpha(t)\varphi_0 = \varphi_\alpha(t) = \left(u^\alpha(t), u_t^\alpha(t)\right), \quad t \ge 0,$$

here u^α is the energy solution of problem (1). By Theorem 2.1, the family of operators $\{S_\alpha(t)\}_{t\ge 0}$ forms a semigroup on X_1.
By Theorem 3.2, the dynamical system $(S_\alpha(t), X_1)$ possesses a global attractor A_α for each $\alpha \in (0,1)$. The key structural stability question is whether the global attractor depends continuously on the dissipation parameter α, which is related to the dissipation limit and the theory of inertial manifolds(see[11]). We pay attention to the family of attractors $\{A_\alpha\}_{\alpha\in(0,1)}$ which are upper semicontinuity at each point $\alpha_0 \in (0,1)$ in the metric of X_1, that is

$$\lim_{\alpha\to\alpha_0} dist_{X_1}\{A_\alpha, A_{\alpha_0}\} = 0, \quad t \ge 0.$$

For the fractional dissipation $(-\Delta)^\alpha u_t$ in model (1), when $\alpha = 0$, it reduces to the classical viscous damping u_t (the weak damping). When $\alpha \in (0,1)$, the gentle dissipation $(-\Delta)^\alpha u_t$ is deliberately vague, while it generally indicates that the damping term does not provide full control of the highest-order spatial derivatives. This stands in contrast to strong damping $\alpha = 2$, where $\Delta^2 u_t$ yields robust compactness to ensure exponential decay of energy. Because the Kirchhoff term $f(\nabla u)$ is not a compact perturbation, it destroys the semilinear structure and requires special compactness argument relying on energy methods.
The damping $(-\Delta)^\alpha u_t, \alpha \in (0,1)$ may fail to provide the smoothing effect needed for exponential attractor compactness. Instead, one must exploit the dispersive term combined with dissipative-compensating estimates. Supercritical nonlinearities $(q > q^*)$ would require new compactness arguments that exploit the interplay between the Kirchhoff term and fractional dissipation more delicately.

Open Problem 2 (Unbounded Domains)
In contrast to the case of bounded domains Ω, the main difficulty lies in how to compensate for the loss of compactness and overcome the restrictions imposed on the growth exponent of nonlinearities. Aiming at this problem, the Kirchhoff-Boussinesq type wave equations on unbounded domains create serious obstacles,

especially focus on the interaction between fractional damping, nonlinear damping and nonlinear force term on the behaviors of solutions and longtime dynamics. By means of classifying the fractional damping, giving full play to the dissipative effect of nonlinear damping and using the smoothing technology. For the standard compactness methods fail in [12] because the energy space $X_1 = V_1 \times V_{-1}$ is not compactly embedded into any larger space. The standard approach to overcome this is to use the tail-estimate method for damped wave equations or the energy equation method for non-compact semiflows.

The tail-estimate method of [13] proceeds to extract a convergent subsequence, then use the uniform tail estimates to extend the convergence to the whole space for every α . The key technical ingredient was a weighted energy estimate. For fractional damping on unbounded domains, the problem remains largely open, the lack of a spectral gap and the nonlocal nature of $(-\Delta)^{\alpha} u_t$ make the tail estimates more delicate, as the dissipation acts in a nonlocal manner. The existence of bi-space attractors is particularly interesting because establish new stability criteria of the $(X_1, X_{1+\alpha})$ attractors on the parameter perturbations.

Open Problem 3 (Non-autonomous)

This is a significant gap for non-autonomous Kirchhoff-Boussinesq type equation

$$u_{tt} + (-\Delta)^{\alpha} u_t + \Delta^2 u - div\left\{\frac{\nabla u}{1+|\nabla u|^2}\right\}$$
$$= \Delta g(u) + f(x,t).$$

Our principal result addresses nonlinearity, time-dependent external forcing $f(x,t)$, wave equation (1) with time-dependent memory kernels or wave equation which phase space is time-dependent. Space of translation bounded functions in $L^2_{loc}(R, V_{-1})$. The longtime dynamics of non-autonomous systems is described by pullback attractors $\{A(t)\}_{t\geq 0}$ instead of global attractors, one can see [14,15], where investigate criterias residual continuity of attractor for perturbation parameters in the time-dependent phase space .

To summarize, the K-B equation with gentle dissipation offers a rich mathematical landscape that bridges nonlinear elasticity, dispersive waves, and infinite-dimensional dynamical systems. While substantial progress has been made, many deep questions remain, particularly in the areas of fractional operators, unbounded domains, and non-autonomous perturbations. We hope this paper will stimulate further research on these fascinating problems.

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