Analysis of Static Friction in Pure Rolling of a Disk

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Abstract: Determining the size and direction of the static friction force at the contact point is still one of the more difficult problems to teach in theoretical mechanics, especially for a disk rolling without slipping on a rough inclined plane. To solve the above problem, we start with the differential equations of planar rigid-body motion and impose the kinematic constraint of no-slip rolling for the disk. Thus, an expression for the static friction force under a combined horizontal force and a couple load on the disk has been obtained. Next, we will study how some main factors influence the direction of friction, such as the size and position of the applied load, the slope of the plane, and the radius of gyration of the disk. The results indicate that pure rolling can be decomposed into two fundamental motion forms: translation of the center of mass and rotation about the center of mass. The role of static friction on these two motion forms is always reciprocal: in pure rolling induced by rotational action, static friction opposes rotation while promoting translation, with its direction being the same as the rolling direction; in pure rolling induced by translational action, static friction opposes translation while promoting rotation, with its direction being opposite to the rolling direction. For complex cases where both actions are present simultaneously, the direction of the resultant static friction can be determined by simplifying all active forces to the center of mass and superposing the individual components.

Keywords: Disk; Pure Rolling; Static Friction; Plane Motion of Rigid Body; Direction Criterion; Theoretical Mechanics Teaching.

1. Introduction

One of the persistent challenges in teaching theoretical mechanics is determining the direction of static friction when a disk (or cylinder or wheel) rolls without slipping on a rough horizontal surface. Students are often unable to solve this problem for two reasons. Firstly, static friction is not a value that can be read directly; it needs to be determined by analysing the motion of the entire body, and pure rolling introduces the problem of coupling translation and rotation [1]. For another, most textbooks simply state that friction opposes sliding but do not elaborate on what happens when one pulls with a force or twists by a couple [2].

Many Solutions have been proposed to this problem. Jiang and Zhao [3] found that for a homogeneous circular wheel, the direction of friction depends on the type of load: only the couple sense is required for a couple alone, and both the point of application and the radius of gyration are needed for a horizontal force. Their work combined rigid-body dynamics and energy methods. Liu and others [4] put forward an energy-based criterion that considers kinetic energy conversion. Other ways include those in Zhen's work [5], which considered multiple load arrangements, and Li and others' [6] direct application of the no-slip condition. Chang and others [7] provided a factor classification, and Hu and others [8] offered an easier two-case rule. Engineering applications have also been added to the body of knowledge by Pang et al. [9].

Despite all the above contributions, there is still no simple rule that students can rely on when they need to determine the direction of friction in a pure rolling problem. Therefore, we will return to the basics and consider only two cases: a disk with a couple and a disk subjected to a horizontal force; from these, we can derive some simple rules that students can actually use.

2. Theoretical Analysis of Friction in Pure Rolling of a Disk

2.1 Fundamental Equations

Pure rolling is defined by the absence of slip at the contact point, which accordingly becomes the instantaneous center of velocity [10]. The friction at this point is static: it acts along the tangent and opposes relative motion between the two surfaces. But in practice, the direction of that relative tendency is rarely clear—it depends on both the external loads and the disk's geometry (radius and radius of gyration). Because pure rolling is a special case of planar rigid-body motion, the static friction can be obtained from the usual rigid-body dynamical equations.

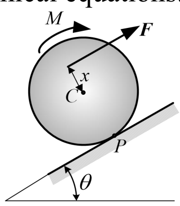


Figure 1. Pure Rolling of a Disk along an Inclined Plane

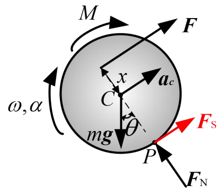


Figure 2. Force and Motion Analysis of a Disk in Pure Rolling Along an Inclined Plane

Consider the system shown in Figure 1: a disk of mass m and radius r undergoes pure rolling along an inclined plane with inclination angle θ. Two loads are applied simultaneously: a force $\boldsymbol{F}$ parallel to the plane and a couple M. The force $\boldsymbol{F}$ acts along a line located at a distance x from the disk's center C, which coincides with both its geometric center and its center of mass. The radius of gyration about C is ρ_C., so the corresponding moment of inertia is $J_C = m\rho_C^2$.

The force analysis and motion analysis of the disk are shown in Figure 2. The contact point P between the disk and the inclined plane is the instantaneous center of velocity. The static friction $\boldsymbol{F}_\mathrm{s}$ is assumed to be in the same direction as the force $\boldsymbol{F}$. The disk moves upward along the inclined plane with angular velocity ω, angular acceleration α, and the acceleration of the center of mass C is $\boldsymbol{a}_C$.

According to the differential equations of plane motion of a rigid body, we have

$$ma_C = F + F_\mathrm{s} - mg\sin\theta \tag{1}$$

$$J_C\alpha = M + Fx - F_\mathrm{s}r \tag{2}$$

The pure rolling condition gives

$$a_C = r\alpha \tag{3}$$

From Eqs. (1) – (3), we have

$$F_\mathrm{s} = \frac{Mr + mg\rho_C^2\sin\theta + Frx - F\rho_C^2}{r^2 + \rho_C^2} \tag{4}$$

2.2 Analysis of Results

It can be seen from Eq. (4) that the static friction on a purely rolling disk on an incline is governed by a combination of parameters: the magnitudes of $\boldsymbol{F}$ and M, the position x at which $\boldsymbol{F}$ is applied, the slope angle θ, and the disk's inertial properties (r and ρ_C). If $Mr + mg\rho_C^2\sin\theta + Frx > F\rho_C^2$, then $\boldsymbol{F}_\mathrm{s}$>0, and the friction acts in the assumed direction; If $Mr + mg\rho_C^2\sin\theta + Frx = F\rho_C^2$, then $\boldsymbol{F}_\mathrm{s}$>0, and no static friction exists; If $Mr + mg\rho_C^2\sin\theta + Frx < F\rho_C^2$, then $\boldsymbol{F}_\mathrm{s}$<0, and the friction acts opposite to the assumed direction. These conditions demonstrate that the friction direction is not intrinsic to the disk or the surface, but rather depends on the specific loading configuration. This sensitivity to multiple factors is precisely what makes the problem difficult for students.

If the disk is homogeneous, then $\rho_C = \frac{\sqrt{2}}{2}r$, and Eq. (4) can be simplified to

$$F_\mathrm{s} = \frac{2M + mgr\sin\theta + 2Fx - Fr}{3r} \tag{5}$$

To facilitate students' understanding and mastery of the methods for determining the direction of static friction, several typical cases of pure rolling of a homogeneous disk are analyzed in detail below.

2.3 Discussion of Special Cases

2.3.1 Homogeneous Disk in Pure Rolling Down an Inclined Plane Under Its Own Weight Only

When M=0 and F=0, that is no external loads other than its own weight act on the disk, the disk rolls freely down the inclined plane, as shown in Figure 3. From Eq. (5), we obtain

$$F_\mathrm{s} = \frac{mg\sin\theta}{3} > 0$$

Since F_s is always positive, the actual direction of static friction is consistent with the assumed direction, that is, upward along the inclined plane and opposite to the direction of motion of the disk. This is precisely the classical conclusion in the problem of pure rolling on an inclined plane: static friction provides the torque necessary for rotation. Specifically, it exerts a counterclockwise torque about the center of mass, causing the disk to acquire angular acceleration and thereby sustaining the pure rolling condition.

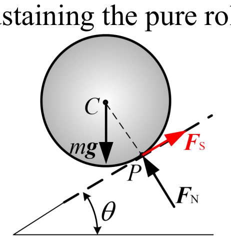


Figure 3. Pure Rolling of a Disk Down an Inclined Plane Under Its Own Weight

2.3.2 Homogeneous Disk in Pure Rolling on a Horizontal Plane under the Action of a Force and a Couple

When θ=0, the disk undergoes pure rolling on a horizontal plane. From Eq. (5), we obtain

$$F_s = \frac{2M + 2Fx - Fr}{3r} \tag{6}$$

(1) Under a couple only (F=0)

When only a couple M is applied, Eq. (6) gives

$$F_s = \frac{2M}{3r} \tag{7}$$

In this case, the pure rolling of the disk is induced by the rotational action (the couple), so the direction of static friction depends on the sense of the couple M. If all external loads are reduced to the center of mass C, the static friction produces a moment about C (the center of rotation [11]) opposite to the external couple, thereby hindering the rotation of the disk, while promoting its translation. Alternatively, if all external loads are reduced to the instantaneous center of velocity P (the rolling center), the static friction must act in the same direction as the rolling motion to ensure no relative sliding at the instantaneous center. In summary, in pure rolling induced by rotational action, static friction hinders rotation and promotes translation, and its direction is the same as the rolling direction, as shown in Figure 4.

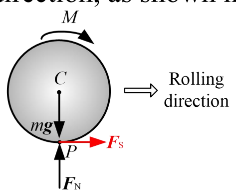


Figure 4. Pure Rolling of a Disk on a Horizontal Plane under the Action of a Couple

(2) Under a horizontal force only

1) Horizontal force acting at the center of mass (x=0)

When M=0 and x=0, i.e., when only a horizontal force F acts at the center of mass, from Eq. (6) we obtain

$$F_s = -\frac{F}{3} \tag{8}$$

In this case, the pure rolling of the disk is induced by the translational action (the horizontal force), and the direction of static friction depends on the direction of $\boldsymbol{F}$. If all external loads are reduced to the center of mass C, the static friction produces a moment about C (the center of rotation) that promotes the rolling of the disk, while hindering its translation. Alternatively, if all external loads are reduced to the instantaneous center of velocity P (the rolling center), the static friction must act in the direction opposite to the rolling motion in order to ensure no relative sliding at the instantaneous center, as shown in Figure 5(a). In summary, in pure rolling induced by translational action, static friction hinders translation and promotes rotation, and its direction is opposite to the rolling direction.

2) Horizontal force acting away from the center of mass ($x \neq 0$)

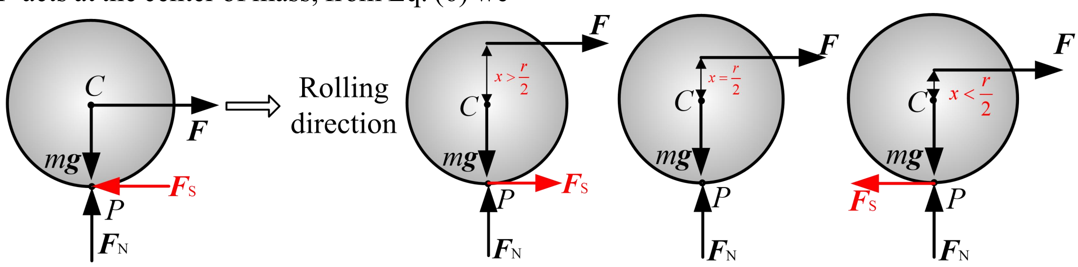
When M=0 and $x \neq 0$, that is, when only a horizontal force F acts at a point other than the center of mass, Eq. (6) gives

$$F_s = \frac{F}{3}\left(\frac{2x}{r} - 1\right) \tag{9}$$

When $x>r/2$, $F_s>0$, and the static friction is in the same direction as $\boldsymbol{F}$, as shown in Figure 5(b).

When $x=r/2$, $F_s=0$, as shown in Figure 5(c). This result has important pedagogical significance: regardless of whether the horizontal plane is smooth or rough, the disk can maintain pure rolling, and its motion state is not affected by friction. The reason for this phenomenon is that the moment of the force F about the center of mass exactly provides the entire rotational effect required for pure rolling, so that no static friction is needed.

When $x<r/2$, $F_s<0$, and the static friction is opposite to $\boldsymbol{F}$, as shown in Figure 5(d).



(a) x= 0 (b) $x>r/2$ (c) $x=r/2$ (d) $x<r/2$

Figure 5. Pure Rolling of a Disk on a Horizontal Plane under the Action of a Horizontal Force

For a homogeneous disk in pure rolling under a horizontal force only, the physical essence of static friction can be further revealed by the superposition method. By reducing the force F to the center of mass C, we obtain a force F' and an induced couple $M'=Fx$. From Eq. (8), the force

$\boldsymbol{F}'$ induces a static friction

$$F_{s1}=-\frac{F'}{3}=-\frac{F}{3}$$

opposite to the rolling direction; from Eq. (7), the induced couple M' induces a static friction

$$F_{s2}=\frac{2M_C}{3r}=\frac{2Fx}{3r}$$

in the same direction as the rolling direction, as shown in Figure 6. Superposing the two components yields

$$F_s=F_{s1}+F_{s2}=-\frac{F}{3}+\frac{2Fx}{3r}=\frac{F}{3}(\frac{2x}{r}-1)$$

This result is exactly consistent with Eq. (9). When $x>r/2$, the same-direction static friction component induced by the induced couple dominates, and the total friction is in the same direction as F. When $x=r/2$, the two effects exactly cancel each other out, and the static friction is zero. When $x<r/2$, the translational effect dominates, and the total friction is opposite to F. The pedagogical advantage of the superposition method lies in decomposing a complex mechanical problem into the linear superposition of two fundamental effects, which facilitates students' understanding of the direction criterion of static friction from a physical perspective.

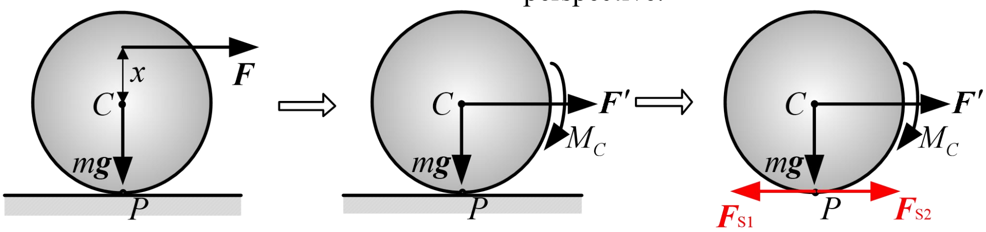
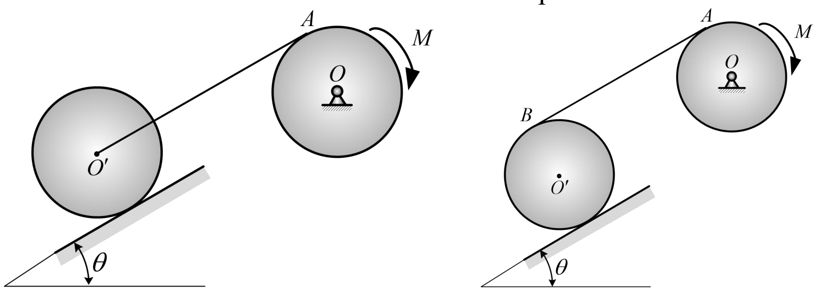


Figure 6. Analysis of Pure Rolling of a Disk on a Horizontal Plane under a Horizontal Force

3. Method for Determining the Direction of Static Friction in Pure Rolling

Based on the foregoing analysis, the following criteria for determining the direction of static friction can be summarized: in pure rolling induced by rotational action, the direction of static friction is the same as the rolling direction; in pure rolling induced by translational action, the direction of static friction is opposite to the rolling direction. In cases where translational and rotational effects coexist, the direction of the resultant static friction must be evaluated by taking both contributions into account.

The application of these criteria can be illustrated through two classic problems of hoisting a cylinder up an inclined plane, adapted from Ref. [7]. In the two systems depicted in Figure 7, the segments $O'A$ and AB are both parallel to the inclined plane, and the friction coefficient between the cylinder and the plane is f. A driving couple M is applied to the fixed pulley O raises the cylinder O', which rolls without slipping upward along the incline throughout the motion. The question is to determine, for each configuration, the direction of static friction acting at the contact between the cylinder and the inclined plane.



(a) Inclined-plane lifting case 1 (b) Inclined-plane lifting case 2

Figure 7. Two Cases of Lifting a Cylinder up an Inclined Plane

Analysis of Figure 7(a): The cylinder O' is pulled at its center by a rope passing over a fixed pulley. At the center of mass O', the forces acting are gravity and the rope tension. Because the tension overcomes the down-slope component of gravity, the cylinder's motion up the incline is driven by a translational action. The static friction therefore opposes the direction of translation-that is, it acts downward along the inclined plane, opposite to the rolling direction.

Analysis of Figure 7(b): The rope is attached to the cylinder at point B. By translating the tension F_T acting at B to the center of mass O', we obtain a force $\boldsymbol{F}_T'$ passing through the center of mass and an induced couple $M_T=F_T\cdot r$. The induced couple generates a static friction component in the rolling direction (upward along the incline), while the force F_T' (together with the down-slope component of gravity) generates a component opposing the rolling direction (downward along the incline). Since the rotational contribution prevails, the net static friction acts in the rolling direction, i.e., upward along the inclined plane.

Comparison of the above two cases reveals that even though the rolling directions of the cylinder in both systems are the same, the directions of static friction can be completely opposite due to different results of reducing the external loads. When analyzing such problems, one should follow the systematic procedure of "reducing to the center of mass → identifying the fundamental type of action → superposing the individual components → determining the resultant direction" rather than relying on intuitive judgment alone.

4. Conclusions

This paper has presented a systematic theoretical analysis of pure rolling of a disk under the combined or separate action of a couple and a force. Our analysis rests on two pillars: the differential equations of planar rigid-body motion and the no-slip constraint that defines pure rolling. The primary results are as follows.

(1) For a disk in pure rolling, the friction at the contact is static. The planar rigid-body equations, together with the no-slip constraint, fix both its magnitude and its direction. It is not fixed; rather, it changes based on the load (how large and where it acts), the slope, and the radius of gyration of the disk. Disks are a special case of round bodies, so the same logic applies to cylinders, rings and cylindrical shells.

(2) Pure rolling can be viewed as a combination of two basic motions: the disk moves as a whole and rotates about its centre. Static friction is a constraint force that opposes the two motions in the above way. When rotation is the cause of motion, friction occurs in the direction of rolling; thus, rotation is reduced but the disk moves forward. When translation is the driving force, friction occurs in the direction opposite to rolling; thus, translation is reduced and the disk rotates. The two effects are to determine the direction of friction in a problem of pure rolling.

(3) When both translation and rotation occur, it is much simpler to analyse by moving all active forces to the centre of mass. The contributions of the resulting principal force and principal couple can then be calculated separately and added up. The direction of the net friction force depends on which contribution is dominant: if the principal force prevails, friction opposes rolling; if the principal couple is dominant, friction aids rolling; and if the two are equal, friction is zero. For a homogeneous disk, this cancellation occurs when the horizontal force is applied at a distance $x=r/2$ from the centre; it is a result worth teaching.

The method we have presented is not arbitrary; it can be derived directly from the standard rigid-body dynamics, and each step in the derivation is explicit and easy to follow. The criteria we have come to be on are simple to state, based on clear physical reasons, and easy to apply in practice. We believe that this work will provide a good teaching material for instructors and students learning about pure rolling problems in theoretical mechanics.

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