Applications and Research Progress of Max–Min Fuzzy Relation Inequalities

Shubin Chen, Xiaopeng Yang
School of Mathematics and Statistics, Hanshan Normal University, Chaozhou, Guangdong, China

Abstract: Max–min fuzzy relation inequalities (max–min FRI) provide a mathematical framework for describing uncertain associations among demands, resources, and service capacities. This review traces the development of max–min FRI from basic solution structures to lexicographically minimal solutions, interval solutions, upper-bounded minimal solutions, and multi-objective optimization. Educational information resource allocation and P2P educational resource sharing are the main application contexts. The review highlights the distinction between Pareto-minimal solutions under upper-bound constraints and global resource minimization, and examines the transition from point solutions to solution-set descriptions. It also identifies current gaps in dynamic parameter updating, multi-objective trade-offs, and secondary selection of upper-bounded minimal solutions. The analysis indicates that max–min FRI has a solid foundation for educational resource allocation, but further work is needed to develop dynamic, robust, and practically implementable scheduling methods.

Keywords: Max–Min Fuzzy Relation Inequality; Minimal Solution; Upper-Bound Constraint; Pareto Optimality; Educational Information Resource; P2P Network; Resource Allocation

1. Introduction

Supply–demand relationships in real-world resource systems often exhibit non-deterministic characteristics. In educational information resources, for example, the service contribution of the same resource node to different user groups varies, and in P2P networks the transmission capacity and content matching degree of each terminal node are difficult to characterize with binary relations. Fuzzy set theory provides a mathematical foundation for describing such graded associations [1]. Since Sanchez proposed composite fuzzy relation equations, the solution and optimization of fuzzy relation equations and fuzzy relation inequalities (FRI) have become an important branch of fuzzy optimization [2,3].

Compared with early research focusing on "whether a solution exists," the research focus in this field has gradually shifted in recent years toward fine-grained characterization of solution-set structures, special solutions under extended constraints, and multi-objective optimization frameworks. In particular, in the study of max–min fuzzy relation inequalities (max–min FRI), scholars have successively proposed concepts such as approximate solutions for inconsistent systems [4]. lexicographically minimal solutions [5]. Wide interval solutions [6]. And minimal solutions under a given upper bound [7]. For scenarios of P2P educational information resource sharing and teaching resource scheduling. These works show that max–min FRI is no longer a pure mathematical abstraction, but can be directly mapped to management problems such as demand satisfaction, capacity constraints, and priority scheduling in educational resource allocation.

However, existing reviews have largely presented these studies in chronological order, with insufficient systematic analysis of the interplay between the evolution of the underlying theory and its application scenarios. More importantly, research on max–min FRI in educational information resource scheduling still remains at the level of static models and single-objective optimization, with insufficient discussion of key issues such as dynamic demand changes, multi-objective trade-offs, and the Pareto nature of minimal solutions under upper-bound constraints. In view of this, this paper takes max–min fuzzy relation inequalities as the theoretical main line and educational information resource allocation and P2P sharing as the core application scenarios, systematically reviewing their model evolution, solution algorithms, and research trends, and clarifying the gaps between current theory and educational practice, in order to provide operable entry points for subsequent research.

2. Related Research Progress

Research on fuzzy relation inequalities (FRI) began with solvability discussions of fuzzy relation equations (FRE). In 1976, Sanchez first proposed the max–min $(\vee, \wedge)$ fuzzy relation equation and applied it to medical diagnostic fuzzy reasoning [2]. Thereafter, Miyakoshi et al. extended the max–min and max–product operators to the continuous max–t-norm framework, proving that the solution sets of such FRE can all be expressed as unions of finitely many closed intervals, and that a unique maximal solution and finitely many minimal solutions exist[8,9]. Drewniak formally proposed the concept of fuzzy relation inequalities (FRI) in 1984[10]; subsequently, related research further developed the lattice structure, path structure, and minimal-solution characterization methods of fuzzy relation inequalities [11]. The above work laid the theoretical foundation for the solution-set structure of FRI: the solution set of a consistent system is spanned by a unique maximal solution (or minimal solution) together with finitely many minimal solutions, exhibiting typical lattice-theoretic features.

2.1 From Max–Min to Multivariate Operators: Commonalities and Distinctions in Solution Structures

Following the systematic achievements within the $max-min$ framework, scholars have gradually extended their research to other fuzzy operators. Bourke *et al.* systematically investigated the solution algorithms for $max-product(\vee, \cdot) FRE$ [12]; Yang *et al.* studied the $min-max$ programming problem subject to $max-product$ fuzzy relational equation constraints, laying a foundation for subsequent optimization under fuzzy relational constraints [13]. It is noteworthy that although $max-t-norm$ -family operators (including $max-min, max-product$) exhibit similarities in solution-set structure—all representable as unions of finitely many closed intervals—non-$max-t-norm$ operators such as $max-plus(\vee, +), max-average$ have also been proved to possess analogous interval-union characteristics [14,15].

In recent years, Guo *et al.* [16] and Shu *et al.* [17] shifted the research perspective toward $min-product(\wedge, \cdot)$ and $min-plus(\wedge, +)$ fuzzy relational inequalities, discovering that although their solution sets still constitute unions of closed intervals, the endpoint properties of these intervals undergo a fundamental change: $min-t-norm$ -type FRIs take a unique minimal solution as the left endpoint and finitely many maximal solutions as the right endpoints, forming an order dual to their $max-t-norm$-type counterparts. This discovery broadens the operator spectrum of fuzzy relational systems and provides new modeling tools for "lower-bound constraint" scenarios such as supply-chain price transmission.

2.2 Special Solutions and Extended Models: From Existence to Structural Refinement

As the foundational solvability theory matured, the research focus gradually shifted from "does a solution exist?" to "fine-grained solution structure" and "special solutions under extended constraints."

(1) Bipolarity and weighted extensions.

Real-world decisions often involve both positive and negative factors simultaneously. Freson *et al.* investigated a linear optimization model with bipolar $max-min$ constraints, laying a foundation for the development of bipolar fuzzy relational models [18]. Subsequent studies extended bipolar models to $max-product, Łukasiewicz$ and general $max-t-norm$ settings [19, 20, 21]. On the other hand, in recent years $max-min$ FRIs have been further extended to weighted $min-max$ optimization and weighted multiple-combination forms; Molai established a $max-min-product$ hybrid FRI and solved the $min-max$ programming under its constraints [22]. These extensions break the implicit "homogeneous components" assumption of classical models, enabling fuzzy relational systems to express more complex decision preferences.

(2) Missing-coefficient systems.

Supply chains and communication networks frequently suffer node failures or link interruptions in practice. Qiu et al. introduced a missing-coefficient model for $max-product$ FRIs, discussing consistency and optimization under arbitrary coefficient-missing scenarios [23]; Yang et al. investigated variable-missing $max-min$ FRIs, proving the necessary and sufficient conditions for system consistency and the solution-set structure when any l variables are missing from the constraints [24]. Recent research has further generalized missing-coefficient models to supply-chain systems, characterizing the complete solution set under missing constraints via $(l+1)$-dimensional paths and characteristic matrices. These works provide a theoretical basis for network reliability analysis.

(3) Special solutions and interval solutions.

Centered on stability analysis, existing research has been extended from the perspective of bilateral fuzzy relational systems [25]; Chen et al. investigated the maximal interval solution and the widest interval solution under the ℓ_1 norm for $max - min$ FRIs [26, 27]. In addition, Chen et al. studied minimal solutions under a given upper-bound constraint for $max - min$ FRIs, pointing out that they are essentially Pareto-minimal elements under upper-bound constraints rather than globally minimal total resources, and presented stepwise algorithms of $O(mn)$ and $O(mn n^2)$ [7]. These works indicate that solution research is deepening from "points" to "intervals" and from "existence" to "robustness." Studies from 2024–2026 further address variable missing, weighted combinations, concentrated solutions, maximal-width interval solutions, and sensitivity analysis [28].

It should be particularly emphasized that the "upper-bound minimal solution" defined by Chen et al. [7] is a minimal element under the Pareto partial-order structure, whose existence depends on the interplay between upper-bound constraints and demand constraints. The establishment of this concept marks a deepening of $max - min$ FRI research from "whether a feasible solution exists" to "how to optimally compress resources at the constraint boundary."

2.3 Optimization Problems: Evolution from Single-Objective to Multi-Objective

Optimization problems under FRI constraints serve as a bridge connecting theoretical research with practical applications. In 1991, Wang *et al.* first proposed fuzzy relational programming with $max - min$ FRE constraints [11]. Fang and Li introduced a linear objective function into $max - min$ FRE constraints, transforming the problem into 0–1 integer programming and solving it by branch-and-bound [29]. Since then, this framework has been widely applied to various fuzzy relational constraints such as $max - product, max - t - norm$ and $min - product$ [30, 31, 32].

For nonlinear objectives, existing studies have further discussed weighted $min - max$ programming under $max - product$ and other fuzzy relational constraints [13, 28]. In the $min - product$ supply-chain model, scholars have further investigated lexicographic maximal pricing [33] as well as multi-objective profit optimization. In particular, lexicographic optimization maximizes prices sequentially according to preset supplier priorities, avoiding the complex trade-offs of multi-objective conflicts and possessing clear managerial significance in supply-chain fixed-priority pricing [33]. However, lexicographic solutions are extremely sensitive to priority ordering and provide only a single solution rather than solution-set information, which limits their applicability when multi-dimensional fairness trade-offs are required.

In recent years, multi-objective optimization has gradually attracted attention. Under the $min - product$ supply-chain framework [34], researchers established a bi-objective model that simultaneously maximizes total profit and the minimum supplier profit, proved that Pareto optimal solutions must be maximal solutions, and designed an improved NSGA-II algorithm to solve the Pareto frontier. However, multi-objective optimization research under $max - min$ FRI constraints remains relatively scarce in the existing literature, especially studies combining the dual requirements of "fairness" and "efficiency" in educational resource sharing, which still await in-depth investigation. The root cause lies in the lattice-structure complexity of the $max - min$ FRI solution set, which makes the analytical characterization of the Pareto frontier difficult: the number of minimal solutions grows exponentially with problem scale, and the dominance relations among minimal solutions lack explicit expressions, leading to high computational overhead for evolutionary algorithms during encoding and fitness evaluation.

2.4 Approximate Solutions and Incompatible Systems

In practical applications, FRI systems are often incompatible due to excessively strong demands or insufficient resources. Pedrycz first proposed a quasi-Newton method for approximate solutions of $max - min$ FREs [35]; Baaj provided exact formulas for approximate solutions under Chebyshev distance for three typical classes of incompatible $max - t - norm$ FREs [36]. Lin and Yang designed approximate-solution algorithms under Chebyshev distance and ℓ_1 norm, respectively; the core idea in both cases is to fine-tune the right-hand-side vector to render the system consistent [37, 38]. Xiao *et al.* addressed inconsistent $max - min$ fuzzy relational equations in teaching information resource sharing, designing a linear search method that minimizes the maximal dissatisfaction degree [4]. These studies provide theoretical tools for compromise decision-making when demands cannot be fully satisfied.

From a methodological perspective, the above approximate-solution studies can be divided into two categories: first, right-hand-side correction based on matrix perturbation, and second, satisfaction optimization based on objective relaxation [4]. The former preserves the semantic integrity of the fuzzy association matrix A, and is applicable when A has clear physical meaning; the latter directly targets domains such as educational management where demand elasticity is relatively large, transforming hard constraints into soft optimization objectives by introducing a dissatisfaction function.

2.5 Review of Existing Research

In view of the above progress, research on $max - min$ fuzzy relational inequalities has formed a relatively complete theoretical spectrum: in terms of operators, extending from $max - min$ to $max - product, min - product$ and general $t - norm/s - norm$; in terms of solution structure, from a single feasible solution to the complete solution set, and further to interval solutions and special solutions; in terms of optimization problems, from single-objective linear programming to lexicographic optimization, and then to multi-objective evolutionary optimization; in terms of application scenarios, from medical diagnosis to P2P networks, supply chains, and educational resource allocation.

Nevertheless, existing research still exhibits the following shortcomings. First, studies on $max - min$ FRIs in educational resource sharing have mostly focused on static models and point solutions, with limited discussion on solution-set structure (e.g., wide-interval solutions) and approximate solutions under dynamic demands. Second, although the $min - product$ framework has yielded rich results in supply-chain optimization, the corresponding educational-resource scheduling model under the $max - min$ framework—especially minimal solutions with upper-bound constraints and multi-objective trade-offs—has not yet been adequately developed. Third, existing literature tends to confuse the notion of "minimal solution" with "globally minimal resources," and the Pareto nature of minimal solutions under upper-bound constraints as well as their algorithmic complexity still need to be further clarified in application-oriented research.

It is particularly worth emphasizing the boundary of computational complexity and tractability. Enumerating all minimal solutions of a $max - min$ FRI is computationally equivalent to the set-covering problem and is NP-hard [6]. Chen *et al.* [7] proved that finding a single Pareto-minimal solution under upper-bound constraints can be accomplished in $O(mnn^2)$ time, providing a computability guarantee for real-time decision-making in large-scale educational networks; however, when one needs to enumerate all upper-bound minimal solutions to support multi-objective secondary selection, the problem reverts to exponential complexity. Existing approximation algorithms [39] mostly provide performance guarantees under Chebyshev or ℓ_1 norm, yet lack explicit connections to classical set-covering approximation algorithms and do not provide lower bounds on the approximation ratio achievable in polynomial time. For educational cloud platforms at the scale of tens of thousands of nodes, how to design fast algorithms with constant approximation ratios while guaranteeing demand satisfaction rates remains an open problem.

3. Typical Application Analysis

The theoretical evolution discussed in Section 2 indicates that research on max-min FRI has progressed from feasibility determination to characterization of solution structures and optimization and selection under constraints. This section connects this theoretical trajectory with specific application scenarios of educational information resource allocation and analyzes the applicability and limitations of existing models in practical applications.

3.1 Educational Information Resource

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Allocation: Direct Mapping and Static Limitations of the Max–Min Framework

In an educational information resource system, let $A = (a_{ij})m\times n$ be the fuzzy association matrix, where $a_{ij} \in [0,1]$ denotes the matching degree between the i-th teaching demand object (e.g., course, class, or knowledge module) and the j-th resource node (e.g., server, terminal, or content repository); let $x = (x_1,...,x_n)^T$ be the resource investment vector, representing the service capacity configuration of each node; and let $\boldsymbol{b}$ be the minimum demand vector. Then the max–min composition $(A \circ x)_i = \max_{1\leq j\leq n} \min(a_{ij}, x_j)$ characterizes "the service level that demand i obtains through its best-matching resource node," while the constraint $A \circ x \geq b$ requires that the minimum service threshold of all demand objects must be satisfied.

Xiao et al. [4] were the first to explicitly apply this model to instructional information resource sharing systems, pointing out that when terminal resource distributions are uneven and download demands cannot be simultaneously satisfied, approximate solutions should be sought with minimum maximum dissatisfaction as the objective. This work established the basic mapping of max–min FRI in educational resource scenarios: resource node capacity → variable $\boldsymbol{x}$, supply–demand matching degree → matrix $\boldsymbol{A}$, minimum teaching demand → vector $\boldsymbol{b}$.

However, educational resource allocation often faces hard constraints on service capacity upper bounds. Each resource node has physical upper limits on storage capacity, bandwidth, or concurrent processing capability, denoted $\bar{\boldsymbol{x}}$; therefore the actual model should be: $A \circ x \geq b$, $x \leq \bar{x}$.

Chen et al. [7] addressed such max–min FRI systems with upper-bound constraints, proposing existence determination and stepwise solution algorithms for upper-bounded minimal solutions. It needs to be particularly emphasized that the "upper-bounded minimal solution" defined in that paper is a Pareto-optimal minimal element under the given upper-bound constraint, i.e., without exceeding the service capacity upper bound, no resource node's investment can be reduced without violating demand satisfaction or exceeding the upper bound. This means:

(1) Upper-bounded minimal solution ≠ global total resource minimization: it reflects the local incompressibility of each resource investment under the $\bar{x}$ constraint, rather than the optimality of total system energy consumption or total cost;

(2) Multi-solution coexistence: due to the existence of the partial-order structure, upper-bounded minimal solutions are usually not unique, and secondary screening using priorities, cost functions, or fairness criteria is required.

This Pareto nature has important implications for educational resource scheduling: in actual management, there is a conflict between "compressing resource investment" and "guaranteeing teaching demand" that cannot be simply aggregated; decision-makers need to clarify whether they are pursuing "minimum investment of a certain resource node" or "minimum total system cost," otherwise directly taking upper-bounded minimal solutions as the "optimal configuration" may lead to goal misalignment.

3.2 P2P Educational Information Resource Sharing: The Cognitive Leap from Point Solutions to Solution-Set Structures

P2P educational information resource sharing is another typical application scenario of max–min FRI, and its research trajectory clearly reflects the cognitive deepening from "finding feasible solutions" to "understanding solution-set structures."

(1) Approximate solutions for inconsistent systems.

In P2P networks, because nodes dynamically join and leave, bandwidth fluctuates, or content is scarce, the demand vector $\boldsymbol{b}$ may be too high and cause the system $A \circ x \geq b$ to be inconsistent. Xiao et al. [4] addressed such situations, designing a linear search method with minimum maximum dissatisfaction as the objective to find approximate solutions. This method essentially restores system consistency by relaxing the right-hand-side vector $\boldsymbol{b}$, providing a quantitative tool for "best-effort" resource scheduling.

(2) Lexicographically minimal solutions and priority scheduling.

When the system is consistent and multiple feasible solutions exist, how to select a "most fair" or "most economical" configuration becomes a key issue. Ma et al. [5] introduced the concept of lexicographically minimal solutions, sequentially minimizing each node's resource investment according to preset resource-node priority orders (e.g., prioritizing remote-area nodes or high-concurrency course nodes). This method's computational complexity is significantly lower than enumerating all minimal solutions, and is particularly suitable for educational resource distribution scenarios in P2P systems where node priorities are clear. However, lexicographic solutions are extremely sensitive to priority ordering and provide only a single solution rather than solution-set information, which is limited when multi-dimensional fairness needs to be balanced.

(3) Interval solutions and the computational-complexity bottleneck.

Chen et al. [6] further pointed out that fully characterizing the solution-set structure of max–min FRI requires finding all minimal solutions, and this problem is computationally equivalent to the set-covering problem, i.e., NP-hard [6]. Therefore, in practically large-scale P2P educational networks, enumerating all minimal solutions is neither realistic nor necessary. As an alternative, Chen et al. proposed the concept of wide interval solutions, characterizing the feasible range of resource allocation by determining the upper and lower bound intervals of the solution set, providing a compact description for resource reservation and robust configuration [26, 27]. This idea is particularly applicable to capacity planning of educational cloud platforms: managers do not need to know every precise feasible solution, but only need to ensure that resource scheduling falls within a safe interval.

In summary, the research chain in the P2P educational information resource sharing scenario can be summarized as: when inconsistent, seek approximate solutions → when consistent, select lexicographic solutions by priority → when robustness is needed, seek interval solutions → when a clear upper bound exists, seek upper-bounded minimal solutions. This chain is not yet closed: how to update interval solutions in real time under dynamic node changes, and how to select upper-bounded minimal solutions under multiple objectives (such as transmission delay, load balancing, and energy consumption), remain open problems.

3.3 Modeling Gaps and Reverse Insights for Cross-Scenario Migration

Compared with educational resource scenarios, whether supply chains and energy systems can directly adopt the max–min FRI model requires careful evaluation. In the existing literature, min–product FRI has achieved mature application in supply-chain price-transmission modeling [17, 33], but research on max–min FRI in these two scenarios remains weak. The fundamental reason lies in the difference of constraint structures of the modeling objects: supply chains involve hard constraints such as delivery times, inventory capacities, and transportation costs, while energy systems are governed by physical laws such as power-flow balance and Kirchhoff's laws, all of which far exceed the expressive power of the max–min composition framework.

A feasible integration path is to use max–min FRI as a fuzzy demand-satisfaction layer coupled with classical network-flow or optimal power-flow models: the upper layer uses $A \circ x \geq b$ to characterize the fuzzy satisfaction threshold of each node, while the lower layer handles concrete inventory–transportation or power-distribution optimization. However, such bi-level coupling has not yet formed a unified theoretical framework of "resource allocation under fuzzy relation constraints." A notable feature of existing research is that educational/P2P scenarios focus on solution-set structure characterization of max–min FRI, while supply-chain scenarios more often adopt min–product FRI to characterize price transmission; the two have systematic differences in solution-set properties (max–min seeks minimal solutions vs. min–product seeks maximal solutions) and optimization objectives.

The reverse insight from the above cross-scenario comparison for educational research is that educational information resource scheduling should not simply adopt the price-optimization models of supply chains, but should build upon the Pareto-minimal-solution structure of max–min FRI to develop secondary selection criteria adapted to educational fairness objectives. Although lexicographic optimization [33] and multi-objective evolutionary algorithms [40] in supply-chain research can be borrowed, their "profit maximization" objective functions have essential differences from the "fairness–efficiency" trade-off in educational resource allocation, and direct migration may lead to value-goal misalignment.

4. Conclusion and Outlook

Focusing on max–min fuzzy relation inequalities, this paper has systematically reviewed their theoretical evolution and application context in educational information resource allocation and P2P sharing, and supplemented the discussion of

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new progress from 2024–2026 on variable absence, stable intervals, centralized solutions, maximum-width interval solutions, and sensitivity analysis. The study shows that this field has gradually developed from early satisfiability determination to refined stages such as solution-set structure characterization, special-solution solving, and constrained optimization. Particularly in educational information resource scenarios, max–min FRI provides a direct mathematical mapping for describing supply–demand matching, demand satisfaction, and resource upper-bound constraints.

However, existing research still has three bottlenecks that urgently need to be broken through.

First, parameter dynamicization and model adaptability. Existing models mostly assume that the fuzzy association matrix $\boldsymbol{A}$, demand vector $\boldsymbol{b}$, and resource upper bound $\bar{\boldsymbol{x}}$ are static parameters. But in actual educational information systems, user access behavior, network topology, and resource quality all evolve over time. How to incorporate time-varying parameters $A(t)$, $\boldsymbol{b(t)}$, $\bar{\boldsymbol{x}}\boldsymbol{(t)}$ into the max–min FRI framework, and design online update algorithms rather than repeated offline solving, is the key for theory to move toward application.

Second, explicit modeling of multi-objective trade-offs. Existing max–min FRI optimization research still mainly focuses on single-objective or specific weighted objectives, lacking systematic characterization of the Pareto front. Drawing on the experience of improved NSGA-II for bi-objective optimization in min–product supply-chain research, establishing an "efficiency–fairness" bi-objective model under the max–min FRI framework and designing structure-aware evolutionary algorithms has clear research value.

Third, secondary selection criteria for upper-bounded minimal solutions. The multi-solution coexistence of upper-bounded minimal solutions means that merely obtaining the set of minimal solutions is insufficient to support final decision-making. Subsequent research should combine specific managerial goals in educational scenarios—such as minimum service cost, load balancing, or priority for key nodes—to establish secondary selection models. For example, one could further solve for the solution that minimizes $\sum c_j x_j$ within the set of upper-bounded minimal solutions, or the solution that minimizes the variance of node loads, thereby forming a two-stage decision process of "feasible solution set → upper-bounded minimal solution set → optimal compromise solution."

In summary, max–min fuzzy relation inequalities have a solid application foundation in educational information resource allocation, but their real transformation into implementable scheduling methods still requires breakthroughs in dynamic modeling, multi-objective optimization, and solution selection mechanisms. Stability and robustness are becoming new research growth points. Future research should not stop at generalized statements such as "introducing artificial intelligence" or "expanding application fields," but should establish a complete research chain from mathematical modeling to algorithm implementation to system verification, targeting dynamic parameters, stable intervals, and multi-objective decision-making.

Acknowledgments

This work was supported by the General Program of the Natural Science Foundation of Guangdong Province (2024A1515010532)

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